Periodicity of High-Order BAM Networks with Piecewise Linear Transfer Function on Time Scales.

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Abstract
This paper presents new results about periodic dynamics of high-order bidirectional associative memory (BAM) networks with piecewise linear transfer (PLT) functions on Time Scales. Invariant sets on time scales are constructed to orientate regions of periodic orbits. By partitions of index sets, we divide BAM networks into two subsystems and obtain a unique periodic trajectory which is exponentially attractive in each invariant set. An example is given to illustrate the theories.

Key words: High-order BAM, invariant sets, periodicity, time scales

1. Introduction

In this paper, we consider the following high-order BAM networks with PLT function on time scale:

\[
\begin{align*}
\dot{x}_i(t) & = -a_i(t)x_i(t) + \sum_{j=1}^{m} \sum_{k=1}^{m} p_{ijk}(t)g\left[y_j(t - \rho(t))\right] \\
& \quad \times g\left[y_k(t - \rho(t))\right] + I_i(t), \quad i \in \mathcal{N} := \{1, 2, \ldots, n\} \\
\dot{y}_j(t) & = -b_j(t)y_j(t) + \sum_{i=1}^{n} \sum_{l=1}^{m} q_{jil}(t)g\left[x_i(t - g(t))\right] \\
& \quad \times g\left[x_l(t - g(t))\right] + J_j(t), \quad j \in \mathcal{M} := \{1, 2, \ldots, m\}
\end{align*}
\]

where \(x_i(t), y_j(t)\) denote the potential of the cell \(i\) and \(j\) at time \(t\), \(a_i(t) > 0\) and \(b_j(t) > 0\) represent the rate of isolation of cells \(i\) and \(j\) from the other states and inputs, respectively. \(p_{ijk}(t), q_{jil}(t)\) are high-order connection weights of neurons. \(I_i(t)\) and \(J_j(t)\) denote the \(i\)th and the \(j\)th component of an external input source introduced from outside the networks to the cell \(i\) and \(j\), respectively. \(\rho(t), g(t)\) are nonnegative bounded functions defined on \(T\). \(T\) is a time scale including zero and \(\sup T = \infty\).

The BAM networks model introduced by [1, 2] is known for its particular feature and good applications in pattern recognition, artificial intelligence, optimization problems and so on. As an extension of Hopfield (recurrent) neural
networks [3, 4], the analysis of the dynamical behaviors is a necessary step for practical design of BAM networks. There are some results of stability for BAM networks with constant delays and distributed delays [5, 6, 8]. As a generalized BAM networks, BAM networks with high-order interconnection weights have been also reported in literature, e.g., [11, 12]. Existence of a unique equilibrium and its stability criteria have been discussed in [13, 14, 17]. Impulsive effects and relative results can be found in [15, 16]. The calculus theory on time scales initiated by S. Hilger [18, 19, 20] has been incorporated into BAM networks [21, 22]. Periodicity and stability analysis of BAM networks on time scales are reported in [23, 24] which can include situations of continuous-time and discrete-time ones [7, 9, 10, 25].

It is wellknown that the network of neurons with PLT functions is a prominent model to emulate the behavior of cortical neurons [26]. The analysis of dynamic properties for networks with PLT functions has attracted growing interest, such as multistability and boundedness[27, 28]. However, not much is known about how the high-order connection strength and PLT functions are related to oscillatory behaviors of BAM on time scales. In this paper, invariant sets have been constructed by some inequalities to locate periodic orbits of BAM on time scales. By using properties of exponential functions, exponential attractivity of periodic trajectories has been also obtained.

The rest of this paper consists of the following sections. Section 2 describes some preliminaries and basic results on time scales. Main results are given in Sections 3. An illustrative example will be given in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminaries

First we recall some basic definitions and results on time scales from [19].

Definition 1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{w \in \mathbb{T} : w > t\}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{w \in \mathbb{T} : w < t\}$. The graininess $\mu : \mathbb{T} \to \mathbb{R}$ is defined by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is said to be right-dense (left-dense) if $\sigma(t) = t$ ($\rho(t) = t$); A point $t \in \mathbb{T}$ is said to be right-scattered (left-scattered) if $\sigma(t) > t$ ($\rho(t) < t$). A function $h : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points and if the left-hand sided limits exists at left-dense points. Denote by $h \in \mathbb{C}_{rd}$.

Definition 2. A time scale $\mathbb{T}$ is said to be $\omega$-periodic if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. Meanwhile, $\sigma(t + n\omega) = \sigma(t) + n\omega$ and $\mu(t + n\omega) = \mu(t)$, $\forall n \in \mathbb{Z}$.

Definition 3. Let $h : \mathbb{T} \to \mathbb{R}$. $h^\Delta(t)$ is said to be the $\Delta$-derivative of $h(t)$ if and only if for any $\epsilon > 0$, there is a neighborhood $\Omega$ of $t$ such that

$$|h(\sigma(t)) - h(s) - h^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|, \forall s \in \Omega.$$
$D^+ h^{\Delta}(t)$ is said to be the Dini derivative of $h(t)$ if, given $\epsilon > 0$, there exists a right neighborhood $\Omega_\epsilon \subset \Omega$ of $t$ such that

$$\frac{h(\sigma(t)) - h(s)}{\sigma(t) - s} < D^+ h^{\Delta}(t) + \epsilon, \ \forall s \in \Omega_\epsilon, \ s > t.$$ 

**Definition 4.** A function $p : \mathbb{T} \to \mathbb{R}$ is said to be a regressive function if and only if $1 + p(t)\mu(t) \neq 0$ for all $t \in \mathbb{T}$.

Let $\mathcal{R}$ be the set of all regressive functions on $\mathbb{T}$, $\mathcal{R}_{rd}$ be the set of all rd-continuous and regressive functions on $\mathbb{T}$ and

$$\mathcal{R}_{rd}^* := \left\{ -p \in \mathcal{R}_{rd}, \ \inf_{s \in \mathbb{T}} \frac{1}{p(s)} > \mu(t), \ t \in \mathbb{T} \right\}.$$

For any $p_1, p_2 \in \mathcal{R}$, define $(p_1 \oplus p_2)(t) = p_1(t) + p_2(t) + \mu(t)p_1(t)p_2(t)$ and $(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)}$, $\forall t \in \mathbb{T}$. We can check that $p_1 \oplus p_2, \ominus p_1 \in \mathcal{R}$. Moreover, $(\mathcal{R}, \oplus)$ is an Abelian regressive group.

**Definition 5.** Let $p \in \mathcal{R}_{rd}$. The exponential function $e_p(t,s)$ is defined on $\mathbb{T}$ by

$$e_p(t,s) = \exp \left[ \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right],$$

where $\xi_h(z)$ is a cylinder transformation with $\xi_0(z) = \log(1+hz)$, $h = 0$.

**Lemma 1.** Let $p \in \mathcal{R}_{rd}$ and $h \in \mathcal{R}_{rd}$, $t_0 \in \mathbb{T}$ and $u_0 \in \mathbb{R}$. The unique solution of initial value problem $u^{\Delta} = p(t)u + h(t)$, $u(t_0) = u_0$, is given by

$$u(t) = e_p(t,t_0)u_0 + \int_{t_0}^t e_p(t,s)h(s)\Delta w.$$

**Property 1.** For any $\omega$-periodic time scale $\mathbb{T}$, if a $\omega$-periodic function $w \in \mathcal{R}_{rd}^*$, then exponential function $e_{-w}(t,s)$ satisfy with: (i) $e_{-w}(t,s) \in (0,1)$ for $t > s$, $t, s \in \mathbb{T}$; (ii) $e_{-w}(t,s) \to 0$ as $t \to +\infty$; (iii) There exist constants $v^1 > 0$ and $v^2 > 0$ such that $e^{-v^1(t-s)} \leq e_{-w}(t,s) \leq e^{-v^2(t-s)}$ for $t > s, t, s \in \mathbb{T}$.

**Proof.** When $\mathbb{T} = \mathbb{R}$, (i)-(iii) are trivial. Consider the case $\mathbb{T} \neq \mathbb{R}$. The assumption $w \in \mathcal{R}_{rd}^*$ implies that $1 - \mu(t)w(t) \in (0,1)$ which leads to $\xi_{\mu(\tau)}(-w(\tau)) < 0$. Hence, (i) holds. By periodicity of $\mu(\cdot)$ and $w(\cdot)$, we have $v := \inf_{\tau \in \mathbb{T}_{[0,\omega)}} \xi_{\mu(\tau)}(-w(\tau)) \in (-\infty, 0)$. Hence $e_{-w}(t,s) \leq \exp(v \int_s^t \Delta \tau) \to 0$ as $t \to +\infty$ which leads to (ii).

Let

$$v^1 := \sup_{\tau \in \mathbb{T}_{[0,\omega]}} \{-\xi_{\mu(\tau)}(-w(\tau))\}, \ v^2 := \inf_{\tau \in \mathbb{T}_{[0,\omega]}} \{-\xi_{\mu(\tau)}(-w(\tau))\}.$$ 

By definition of exponential function, we can check that (iii) holds. ■

For any subset $B \subseteq \mathbb{R}$, set $T_B := \mathbb{T} \cap B$ and $t_B := \sup_{t \in \mathbb{T}} \{\rho(t), \varphi(t)\}$. Let $S(\mathbb{T}_{[-t,0]}, \mathbb{R}^{m+n})$ be the Banach space of all bounded rd-continuous functions mapping $\mathbb{T}_{[-t,0]}$ into $\mathbb{R}^{m+n}$ with sup norm. Let $\ell \in \mathbb{T}_{[0,\omega]}$ and $u(s)$ be a bounded rd-continuous function. Define $u_\ell(s) = u(\ell + s)$, $s \in \mathbb{T}_{[-t,0]}$ and hence
$u_t(r) \in S(T_{[t_D,0]}, \mathbb{R}^{m+n})$. For any initial condition $\psi \in S(T_{[t_D,0]}, \mathbb{R}^{m+n})$, denote by $(x(t; \psi), y(t; \psi))$ the solution of TLBAMs (1).

**Definition 6.** For a given time scale $\mathbb{T}$, a subset $D \subseteq \mathbb{R}^{n \times m}$ is said to be an invariant set of TLBAMs (1) if and only if for any $\psi \in S(T_{[t_D,0]}, D)$ and $t \in T_{[0, +\infty)}$, $(x_t(\psi), y_t(\psi)) \in S(T_{[t_D,0]}, D)$.

**Definition 7.** A pair of $(P_N, N_N)$ is often said to be a partition of index set $N$ if $P_N \cup N_N = N$ and $P_N \cap N_N = \emptyset$.

### 3. Main Results

In this section, we assume $\mathbb{T}$ is a $\omega$-periodic time scale, $a_i(t), b_j(t), I_i(t), q_{ij}(t)$ and $J_j(t)$ are $\omega$-periodic functions on $\mathbb{T}$. The PLT function $g(s) = \max\{0, s\}$. For $G \in \mathbb{C}_{rd}$, define

$$
G := \inf_{t \in T} |G(t)|, \quad G^+(t) := \max\{G(t), 0\},
$$

$$
\bar{G} := \sup_{t \in T} |G(t)|, \quad G^-(t) := \min\{G(t), 0\}.
$$

**Theorem 1.** Assume that $a_i, b_j \in \mathbb{R}_{rd}$ for all $i \in N, j \in M$. For any partition $(P_N, N_N)$ of index set $N$ and $(P_M, N_M)$ of index set $M$, if there exist constants $\alpha_i^1, \beta_j^1$ with $0 < \alpha_i^1 < \beta_j^1$ ($r \in P_N$) and $\alpha_i^2, \beta_j^2$ with $0 < \alpha_i^2 < \beta_j^2$ ($r \in P_M$) such that

$$
(H^{P_N}) : \forall r \in P_N,
\begin{cases}
-a_r(t) \alpha_i^1 + \sum_{j, k \in P_M} p_{rjk}(t) \alpha_j^1 \beta_k^1 + \sum_{j, k \in P_M} p_{rjk}^-(t) \beta_j^1 \beta_k^1 + I_r > 0, \\
-a_r(t) \beta_i^1 + \sum_{j, k \in P_M} p_{rjk}(t) \beta_j^1 \beta_k^1 + \sum_{j, k \in P_M} p_{rjk}^+(t) \alpha_j^1 \beta_k^1 + \bar{I}_r < 0
\end{cases}
$$

and

$$
(H^{P_M}) : \forall r \in P_M,
\begin{cases}
-b_r(t) \alpha_i^1 + \sum_{j, k \in P_N} q_{rjk}(t) \alpha_j^1 \alpha_k^1 + \sum_{j, k \in P_N} q_{rjk}^+(t) \beta_j^1 \beta_k^1 + J_r > 0, \\
-b_r(t) \beta_i^1 + \sum_{j, k \in P_N} q_{rjk}^+(t) \beta_j^1 \beta_k^1 + \sum_{j, k \in P_N} q_{rjk}^-(t) \alpha_j^1 \alpha_k^1 + \bar{J}_r < 0
\end{cases}
$$

hold for $\forall t \in T$,

$$
(H^{N_N}) : \forall l \in N_N, \sum_{j, k \in P_M} \left[ p_{ljk}^+(t) \beta_j^1 \beta_k^1 + p_{ljk}^-(t) \alpha_j^1 \alpha_k^1 \right] + \bar{I}_l < 0
$$

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and
\[
(H^{N_M}) : \forall l \in N_M, \sum_{j, k \in P_N} \left[ q_{l,j,k}^+ (t) \beta^l_j \beta^l_k + p_{l,j,k}^- (t) \alpha^l_j \alpha^l_k \right] + \bar{J}_l < 0
\]

hold for \( \forall t \in \mathbb{T} \), then \( S^N_M \) is an invariant set of TLBAMs (1). Moreover, TLBAMs (1) has one periodic trajectory in \( S^N_M \) and it exponentially attracts all trajectories of \( S^N_M \), where \( S^N_M = S^N_X \times S^N_Y \) and

\[
S^N_X := \left\{ x \in \mathbb{R}^n \mid x_r \in (\alpha^l_r, \beta^l_r), r \in P_N, x_r \in (-\infty, 0), r \in N_N \right\},
\]
\[
S^N_Y := \left\{ y \in \mathbb{R}^m \mid y_r \in (\alpha^l_r, \beta^l_r), r \in P_M, y_r \in (-\infty, 0), r \in N_M \right\}.
\]

Proof. First, we will prove \( S^N_M \) is an invariant set, i.e., given any initial \( \psi = (\psi^1, \psi^2) \in S(\mathbb{T}_{[-T, 0]}, S^N_M), x(t; \psi) \in S^N_X \) and \( y(t; \psi) \in S^N_Y \) for all \( t \in \mathbb{T}_{(0, +\infty)} \). For the components of \( x(t; \psi) \), we consider three cases as follows:

Case 1: There exists a \( r \in P_N \) such that there exists a rd-continuous point \( t^3 \in \mathbb{T}_{(0, +\infty)} \) satisfying \( x_r(t^3) = \alpha_r^l, x_r^\Delta(t^3) \leq 0 \) or there is a right-scattered point \( t^3 \in \mathbb{T}_{(0, +\infty)} \) satisfying \( x_r(t^3) > \alpha_r^l \) and \( x_r(\sigma(t^3)) < \alpha_r^l \).

If \( t^3 \) is a rd-continuous point in \( \mathbb{T}_{(0, +\infty)} \), then it follows from TLBAMs (1) and \( (H^{P_N}) \) that
\[
x_r^\Delta(t^3) \geq -a_r(t^3) \alpha_r^l + \sum_{j, k \in P_M} p^-_{r,j,k}(t^3) \alpha_j^l \alpha_k^l + \sum_{j, k \in P_M} p^+_{r,j,k}(t^3) \alpha_j^l \alpha_k^l + I_r > 0. \tag{2}
\]

If \( t^3 \) is a right-scattered point in \( \mathbb{T}_{(0, +\infty)} \), then it follows from TLBAMs (1) and \( (H^{P_N}) \) that
\[
x_r(\sigma(t^3)) = x_r(t^3) + \mu(t^3) x_r^\Delta(t^3) \geq (1 - \mu(t^3) a_r(t^3)) \alpha_r^l + \mu(t^3) \times \left[ \sum_{j, k \in P_M} p_{r,j,k}(t^3) g(y_j(t^3 - \rho(t^3))) g(y_k(t^3 - \rho(t^3))) + I_r(t^3) \right] \geq \alpha_r^l + \mu(t^3) \left[ -a_r(t^3) \alpha_r^l + \sum_{j, k \in P_M} p^+_{r,j,k}(t^3) \alpha_j^l \alpha_k^l \right. \right. \]
\[
+ \left. \left. \sum_{j, k \in P_M} p^-_{r,j,k}(t^3) \beta_j^l \beta_k^l + I_r \right] > \alpha_r^l. \tag{3}
\]

Obviously, both (2) and (3) contradict Case 1.

Case 2: There exists a \( r \in P_N \) such that there exists a rd-continuous point \( t^3 \in \mathbb{T}_{(0, +\infty)} \) satisfying \( x_r(t^3) = \beta_r^l, x_r^\Delta(t^3) \geq 0 \) or there is a right-scattered point \( t^3 \in \mathbb{T}_{(0, +\infty)} \) satisfying \( x_r(t^3) < \beta_r^l \) and \( x_r(\sigma(t^3)) > \beta_r^l \).
If \( t^8 \) is a rd-continuous point in \( T_{(0, +\infty)} \), then it follows from TLBAMs (1) and \((H_{P_N})\) that

\[
x_r^\Delta (t^8) \leq -a_r(t^8) \beta_r^\Delta + \sum_{j, k \in P_M} p_{rjk}^+ (t) \beta_j^\Delta \beta_k^\Delta + \sum_{j, k \in P_M} p_{rjk}^- (t) \alpha_j^\Delta \alpha_k^\Delta + \bar{I}_r < 0.
\]

(4)

If \( t^8 \) is a right-scattered point in \( T_{(0, +\infty)} \), then it follows from TLBAMs (1) and \((H_{P_N})\) that

\[
x_r (\sigma(t^8)) = x_r(t^8) + \mu(t^8) x_r^\Delta (t^8) \leq (1 - \mu(t^8) a_r(t^8)) \beta_r^\Delta + \mu(t^8) \sum_{j, k \in P_M} p_{rjk}^+ (t^8) g(y_j(t^8) - \rho(t^8)) g(y_k(t^8) - \rho(t^8)) + I_r(t^8)
\]
\[
\leq \beta_r^\Delta + \mu(t^8) \left[ -a_r(t^8) \beta_r^\Delta + \sum_{j, k \in P_M} p_{rjk}^+ (t) \beta_j^\Delta \beta_k^\Delta + \sum_{j, k \in P_M} p_{rjk}^- (t) \alpha_j^\Delta \alpha_k^\Delta + \bar{I}_r \right] < \beta_r^\Delta.
\]

(5)

Both (4) and (5) contradict Case 2.

**Case 3:** There exists a \( l \in N_N \) such that there exists a rd-continuous point \( t^8 \in T_{(0, +\infty)} \) satisfying \( x_l(t^8) = 0 \), \( x_l^\Delta (t^8) \geq 0 \) or there is a right-scattered point \( t^8 \in T_{(0, +\infty)} \) satisfying \( x_l(t^8) < 0 \) and \( x_l (\sigma(t^8)) > 0 \).

If \( t^8 \) is a rd-continuous point in \( T_{(0, +\infty)} \), then it follows from TLBAMs (1) and \((H_{N_N})\) that

\[
x_l^\Delta (t^8) \leq \sum_{j, k \in P_M} \left[ p_{ljk}^+ (t^8) \beta_j^\Delta \beta_k^\Delta + p_{ljk}^- (t^8) \alpha_j^\Delta \alpha_k^\Delta \right] + \bar{I}_l < 0
\]

(6)

If \( t^8 \) is a right-scattered point in \( T_{(0, +\infty)} \), then it follows from TLBAMs (1) and \((H_{N_N})\) that

\[
x_l (\sigma(t^8)) = x_l(t^8) + \mu(t^8) x_l^\Delta (t^8) \leq (1 - \mu(t^8) a_l(t^8)) x_l(t^8)
\]
\[
+ \mu(t^8) \left[ \sum_{j, k \in P_M} p_{ljk}^+ (t^8) g(y_j(t^8) - \rho(t^8)) g(y_k(t^8) - \rho(t^8)) + I_l(t^8) \right]
\]
\[
\leq \mu(t^8) \left[ \sum_{j, k \in P_M} \left[ p_{rjk}^+ (t) \beta_j^\Delta \beta_k^\Delta + p_{rjk}^- (t) \alpha_j^\Delta \alpha_k^\Delta \right] + \bar{I}_l \right] < 0.
\]

(7)

Both (6) and (7) contradict Case 3.

By contradictions of Case 1 to Case 3, we get that \( \alpha_r^\Delta < x_r(t) < \beta_r^\Delta \) for all \( t \in T_{[0, +\infty)} \) if \( r \in P_N \), and \( x_l(t) < 0 \) for all \( t \in T_{[0, +\infty)} \) if \( l \in N_N \); By similar
argument, we can show that $\alpha^l_\beta < y_l(t) < \beta^l_\beta$ for all $t \in \mathbb{T}_{[0, +\infty)}$ if $r \in P_M$, and $y_l(t) < 0$ for all $t \in \mathbb{T}_{[0, +\infty)}$ if $l \in N_M$. Therefore, $\mathcal{S}^N_M$ is an invariant set of TLBAMs (1).

Next, we will prove there exists one exponentially attractive periodic trajectory in $\mathcal{S}^N_M$. Let $(x(t; \psi), y(t; \psi))$ and $(x(t; \hat{\psi}), y(t; \hat{\psi}))$ be any two solutions of TLBAMs (1) with initial conditions $\psi, \hat{\psi} \in \mathcal{S}(\mathbb{T}_{[-t_p, 0]}, \mathcal{S}^N_M)$, respectively. By the invariance of $\mathcal{S}^N_M$, we get that $(x(t; \psi), y(t; \psi)), (x(t; \hat{\psi}), y(t; \hat{\psi})) \in \mathcal{S}^N_M$ for $\forall t \in \mathbb{T}_{[0, +\infty)}$. For $i \in N, j \in M$, denote $u_i(t) := x_i(t, \psi) - x_i(t, \hat{\psi})$ and $v_j(t) := y_j(t, \psi) - y_j(t, \hat{\psi})$. Divide TLBAMs (1) into two subsystems as follows:

\begin{align} 
\begin{cases} 
\left. u_i^\Delta(t) \right|_{i \in P_N} &= -a_i(t)u_i(t) + \sum_{j, k \in P_M} p_{ijk}(t) \left[ v_j(t - \rho(t)) \right. \\
& \left. \times y_k(t - \rho(t); \psi) + y_j(t - \rho(t); \hat{\psi})v_k(t - \rho(t)) \right], \\
\left. v_j^\Delta(t) \right|_{j \in P_M} &= -b_j(t)v_j(t) + \sum_{i, i \in P_N} q_{ji}(t) \left[ u_i(t - \rho(t)) \right. \\
& \left. \times x_i(t - \rho(t); \psi) + x_i(t - \rho(t); \hat{\psi})u_i(t - \rho(t)) \right] 
\end{cases} 
\end{align} 

(S1)

and

\begin{align} 
\begin{cases} 
\left. u_i^\Delta(t) \right|_{i \in N_N} &= -a_i(t)u_i(t) + \sum_{j, k \in P_M} p_{ijk}(t) \left[ v_j(t - \rho(t)) \right. \\
& \left. \times y_k(t - \rho(t); \psi) + y_j(t - \rho(t); \hat{\psi})v_k(t - \rho(t)) \right], \\
\left. v_j^\Delta(t) \right|_{j \in N_M} &= -b_j(t)v_j(t) + \sum_{i, i \in P_N} q_{ji}(t) \left[ u_i(t - \rho(t)) \right. \\
& \left. \times x_i(t - \rho(t); \psi) + x_i(t - \rho(t); \hat{\psi})u_i(t - \rho(t)) \right] 
\end{cases} 
\end{align} 

(S2)

It follows from (S1) that

\begin{align} 
D^+|u_i(t)|_i \in P_N &\leq -a_i(t)|u_i(t)| + \sum_{j, k \in P_M} |p_{ijk}(t)| \\
& \times \left[ \beta^T_k |v_j(t - \rho(t))| + \beta^T_j |v_k(t - \rho(t))| \right] 
\end{align} 

(8)

and

\begin{align} 
D^+|v_j(t)|_j \in P_M &\leq -b_j(t)|v_j(t)| + \sum_{i, i \in P_N} |q_{ji}(t)| \\
& \times \left[ \beta^T_i |u_i(t - \rho(t))| + \beta^T_i |u_i(t - \rho(t))| \right]. 
\end{align} 

(9)
One gets from \((H_\mathcal{N})\) and \((H_\mathcal{M})\) that
\[
\begin{align*}
\sum_{j,k \in \mathcal{P}_\mathcal{N}} \left[ \beta^+_j \beta^-_k - \alpha^+_j \alpha^-_k \right] |p_{rjk}(t)| + \sum_{j \in \mathcal{P}_\mathcal{N}} \left[ \beta^+_j \beta^-_k - \alpha^+_j \alpha^-_k \right] |q_{rjk}(t)| + I_r - J_r < a_r(t)(\beta^+_r - \alpha^+_r), & \quad r \in \mathcal{P}_\mathcal{N} \\
\sum_{j,k \in \mathcal{P}_\mathcal{M}} \left[ \beta^+_j \beta^-_k - \alpha^+_j \alpha^-_k \right] |p_{rjk}(t)| + \sum_{j \in \mathcal{P}_\mathcal{M}} \left[ \beta^+_j \beta^-_k - \alpha^+_j \alpha^-_k \right] |q_{rjk}(t)| + J_r - I_r < b_r(t)(\beta^+_r - \alpha^+_r), & \quad r \in \mathcal{P}_\mathcal{M}
\end{align*}
\]
which leads to
\[
\begin{align}
-a_r(t) + \frac{1}{v^+_r} \sum_{j,k \in \mathcal{P}_\mathcal{M}} |p_{rjk}(t)|(v^+_j \beta^-_k + v^-_j \beta^+_k) < 0, & \quad r \in \mathcal{P}_\mathcal{N} \\
b_r(t) + \frac{1}{v^-_r} \sum_{j,k \in \mathcal{P}_\mathcal{M}} |q_{rjk}(t)|(v^+_j \beta^-_k + v^-_j \beta^+_k) < 0, & \quad r \in \mathcal{P}_\mathcal{M}
\end{align}
\tag{10}
\]
where \(v^+_i := \beta^+_i - \alpha^+_i, r \in \mathcal{P}_\mathcal{N}\) and \(v^-_i := \beta^+_i - \alpha^+_i, r \in \mathcal{P}_\mathcal{M}\). It follows from Property 1 and (10) that there exists a sufficiently small constant \(w^\mathfrak{h}\) with \(0 < w^\mathfrak{h} < \min_{i \in \mathcal{P}_\mathcal{N}, j \in \mathcal{P}_\mathcal{M}} \{a_i, b_j\}\) such that \(1 - w^\mathfrak{h} \mu(t) > 0\) and
\[
\begin{align}
-a_i(t) + w^\mathfrak{h} + \frac{1}{v^+_i} \sum_{j,k \in \mathcal{P}_\mathcal{M}} |p_{ijk}(t)|(v^+_i \beta^-_k + v^-_j \beta^+_j) \\
\times e_{\Theta(-w^\mathfrak{h})}(t, t - \rho(t)) < 0, & \quad i \in \mathcal{P}_\mathcal{N} \\
b_j(t) + w^\mathfrak{h} + \frac{1}{v^-_j} \sum_{i \in \mathcal{P}_\mathcal{N}} |q_{ji}(t)|(v^+_i \beta^-_i + v^-_j \beta^+_j) \\
\times e_{\Theta(-w^\mathfrak{h})}(t, t - \sigma(t)) < 0, & \quad j \in \mathcal{P}_\mathcal{M}
\end{align}
\tag{11}
\]
for all \(\forall t \in \mathbb{T}\).

Define rd-continuous functions \(\hat{u}_i, \hat{v}_j \in \mathbb{C}_{rd}\) by
\[
\hat{u}_i(t) := \frac{e_{\Theta(-w^\mathfrak{h})}(t, 0)}{v^+_i} |u_i(t)|, \quad \hat{v}_j(t) := \frac{e_{\Theta(-w^\mathfrak{h})}(t, 0)}{v^-_j} |v_j(t)|, \quad i \in \mathcal{P}_\mathcal{N}, j \in \mathcal{P}_\mathcal{M}.
\]
It follows from (8) and (9) that
\[
D^+ \hat{u}_i^\Delta(t) \bigg|_{i \in \mathcal{P}_\mathcal{N}} \leq \Theta(-w^\mathfrak{h})(t) e_{\Theta(-w^\mathfrak{h})}(t, 0) \frac{|u_i(t)|}{v^+_i} + e_{\Theta(-w^\mathfrak{h})}(\sigma(t), 0) \frac{D^+|u_i(t)|}{v^+_i} \Delta
\leq \left[ -a_i(t) \left(1 + \Theta(-w^\mathfrak{h})(\mu(t))\right) + \Theta(-w^\mathfrak{h})(t) \right] \hat{u}_i(t)
+ \left(1 + \Theta(-w^\mathfrak{h})(t) \mu(t)\right) \frac{1}{v^+_i} \sum_{j,k \in \mathcal{P}_\mathcal{M}} |p_{ijk}(t)| e_{\Theta(-w^\mathfrak{h})}(t, t - \rho(t))
\times [v^+_j \beta^-_k \hat{v}_j(t - \rho(t)) + v^-_j \beta^+_k \hat{v}_k(t - \rho(t))]
\tag{12}
\]
and
\[
D^+ \hat{v}_j^\Delta(t) \bigg|_{j \in \mathcal{P}_\mathcal{M}} \leq \Theta(-w^\mathfrak{h})(t) e_{\Theta(-w^\mathfrak{h})}(t, 0) \frac{|v_j(t)|}{v^-_j} + e_{\Theta(-w^\mathfrak{h})}(\sigma(t), 0) \frac{1}{v^-_j} D^+|v_j(t)| \Delta
\]

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\[
\begin{align*}
\leq & \left[-b_j(t) \left(1 + \Theta(-w^z)(t)\mu(t)\right) + \Theta(-w^b)(t)\right] \hat{\psi}_j(t) \\
+ & \left(1 + \Theta(-w^z)(t)\mu(t)\right) \frac{1}{v_j} \sum_{i,j \in P_N} |q_{ji}u(t)| e_{\Theta(-w^b)}(t, t - g(t)) \\
\times & \left[v_i^\parallel \beta_i^\parallel \hat{u}_i(t - g(t)) + v_i^\parallel \beta_i^\parallel \hat{u}_i(t - g(t))\right].
\end{align*}
\] (13)

Given any constant \( q > 1 \), we will assert that

\[
\hat{u}_i(t), \hat{v}_j(t) < q\|\psi - \hat{\psi}\| \triangleq \delta, \quad \forall t \in \mathbb{T}_{[t_0, +\infty)}, i \in P_N, j \in P_M.
\] (14)

Clearly, \( \hat{u}_i(s), \hat{v}_j(s) < \delta \) for all \( s \in \mathbb{T}_{[-t_0, t_0]} \). On the contrary, we need two cases to discuss:

**Case a:** There exist some \( t^\delta \in \mathbb{T}_{[t_0, +\infty)} \) and some \( i \in P_N \) such that \( \hat{u}_i(t^\delta) = \delta \) and \( \hat{u}_i^\delta(t^\delta) \geq 0 \). However, it follows from (11), (12) and (14) that

\[
D^+ \hat{u}_i^\delta(t^\delta) \bigg|_{t \in P_N} \leq \frac{-a_i(t^\delta) + w^b}{1 - w^z\mu(t^\delta)} - \frac{1}{1 - w^z\mu(t^\delta)}
\times \frac{1}{v_i} \sum_{j,k \in P_M} |p_{ij}k(t)|(v_j^\parallel \beta_j^\parallel + v_k^\parallel \beta_j^\parallel) e_{\Theta(-w^b)}(t, t - \rho(t)) \delta,
\]

\[
\leq \frac{\delta}{1 - w^z\mu(t^\delta)} \left[-a_i(t^\delta) + w^bight]
\]

\[
+ \frac{1}{v_i} \sum_{j,k \in P_M} |p_{ij}k(t)|(v_j^\parallel \beta_j^\parallel + v_k^\parallel \beta_j^\parallel) e_{\Theta(-w^b)}(t, t - \rho(t)) \delta < 0
\]

which contradicts \( \hat{u}_i^\delta(t^\delta) \geq 0 \).

**Case b:** There exist some \( t^\delta \in \mathbb{T}_{[t_0, +\infty)} \) and some \( j \in P_M \) such that \( \hat{v}_j(t^\delta) = \delta \) and \( \hat{v}_j^\delta(t^\delta) \geq 0 \). However, it follows from (11), (13) and (14) that

\[
D^+ \hat{v}_j^\delta(t^\delta) \bigg|_{t \in P_M} \leq \frac{-b_j(t^\delta) + w^b}{1 - w^z\mu(t^\delta)} - \frac{1}{1 - w^z\mu(t^\delta)}
\times \frac{1}{v_j} \sum_{i,k \in P_N} |q_{ji}u(t)|(v_i^\parallel \beta_i^\parallel + v_i^\parallel \beta_i^\parallel) e_{\Theta(-w^b)}(t, t - \rho(t)) \delta,
\]

\[
\leq \frac{\delta}{1 - w^z\mu(t^\delta)} \left[-b_j(t^\delta) + w^zight]
\]

\[
+ \frac{1}{v_j} \sum_{i,k \in P_N} |q_{ji}u(t)|(v_i^\parallel \beta_i^\parallel + v_i^\parallel \beta_i^\parallel) e_{\Theta(-w^b)}(t, t - \rho(t)) \delta < 0
\]

which contradicts \( \hat{v}_j^\delta(t^\delta) \geq 0 \).
Case a and Case b show that our assertion (14) holds. Let $q \to 1$, we get that

$$|u_i(t)| \leq \ell \| \psi - \hat{\psi} \| e^{-wa}(t, 0), \quad |v_j(t)| \leq \ell \| \psi - \hat{\psi} \| e^{-wa}(t, 0)$$

(15)

for all $i \in P_N$, $j \in P_M$ and $t \in T[0, +\infty)$, where $\ell = \max_{i \in P_N, j \in P_M} \{ v^i, v^j \}$.

Next, we consider subsystem $(S_2)$. It follows from $(S_2)$ and Lemma 1 that

$$
\begin{align*}
|u_i(t)|_{i \in N_N} &= e^{-a_i(t, 0)}u_i(0) + \sum_{j, k \in P_M} \int_0^t p_{ijk}(w)e^{-a_i(t, \sigma(w))} \\
&\quad \times \left[ v_j(w - \rho(w))y_k(w - \rho(w); \psi) \\
&\quad + y_j(w - \rho(w); \hat{\psi})v_k(w - \rho(w)) \right] \Delta w,
\end{align*}

\begin{align*}
|v_j(t)|_{j \in N_M} &= e^{-b_j(t, 0)}v_j(0) + \sum_{i, l \in P_N} \int_0^t q_{jil}(w)e^{-b_j(t, \sigma(w))} \\
&\quad \times \left[ u_i(w - \rho(w))x_l(w - \rho(w); \psi) \\
&\quad + x_i(w - \rho(w); \hat{\psi})u_l(w - \rho(w)) \right] \Delta w
\end{align*}

Together with (15), we get that

$$
\begin{align*}
|u_i(t)|_{i \in N_N} &\leq e^{-a_i(t, 0)}u_i(0) + \sum_{j, k \in P_M} \left[ \int_0^t |p_{ijk}(w)|e^{-a_i(t, \sigma(w))} \right] \\
&\quad \times (\beta_k^3 + \beta_j^3)e^{-wa}(w - \rho(w), 0) \Delta w \ell \| \psi - \hat{\psi} \|,
\end{align*}

\begin{align*}
|v_j(t)|_{j \in N_M} &\leq e^{-b_j(t, 0)}v_j(0) + \sum_{i, l \in P_N} \left[ \int_0^t |q_{jil}(w)|e^{-b_j(t, \sigma(w))} \right] \\
&\quad \times (\beta_i^3 + \beta_l^3)e^{-wa}(w - \rho(w), 0) \Delta w \ell \| \psi - \hat{\psi} \|,
\end{align*}

which leads to

$$
\begin{align*}
|u_i(t)|_{i \in N_N} &\leq \left[ e^{-wa}(t, 0) + \sum_{j, k \in P_M} \int_0^t |p_{ijk}(w)|e^{-wa}(t, \sigma(w)) \right] \\
&\quad \times (\beta_k^3 + \beta_j^3)e^{-wa}(w - \rho(w), 0) \Delta w \ell \| \psi - \hat{\psi} \|,
\end{align*}

\begin{align*}
&\leq \left[ e^{-wa}(t, 0) + \sum_{j, k \in P_M} \int_0^t \frac{|p_{ijk}(w)|}{1 - w^2 \mu(w)}e^{-wa}(t, w) \right]
\end{align*}
\[
\begin{align*}
\times & \quad (\beta_k^+ + \beta_j^\delta) e_{-w^k}(w - \rho(w), 0) \Delta w \bigg\| \psi - \hat{\psi} \bigg\| \\
\leq & \quad \left[ e_{-w^k}(t, 0) + \int_0^t \frac{e_{-w^k}(t, 0)}{(1 - w^2 \mu(w))e_{-w^k}(w, w - \rho(w))} \Delta w \right. \\
\times & \quad \left. \sum_{j, k \in P_M} \tilde{p}_{ijk}(\beta_k^+ + \beta_j^\delta) \right] \bigg\| \psi - \hat{\psi} \bigg\| \quad (16)
\end{align*}
\]

and

\[
\begin{align*}
|v_j(t)|_{j \in N_M} & \leq \left[ e_{-w^k}(t, 0) + \sum_{i, l \in P_N} \int_0^t q_{jil}(w) e_{-w^k}(t, \sigma(w)) \right. \\
\times & \quad \left. (\beta_i^+ + \beta_l^\delta) e_{-w^k}(w - \rho(w), 0) \Delta w \right] \bigg\| \psi - \hat{\psi} \bigg\| \\
\leq & \quad \left[ e_{-w^k}(t, 0) + \sum_{i, l \in P_N} \int_0^t q_{jil}(w) \frac{e_{-w^k}(t, 0)}{1 - w^2 \mu(w)} e_{-w^k}(t, w) \right. \\
\times & \quad \left. (\beta_i^+ + \beta_l^\delta) e_{-w^k}(w - \rho(w), 0) \Delta w \right] \bigg\| \psi - \hat{\psi} \bigg\| \\
\leq & \quad \left[ e_{-w^k}(t, 0) + \int_0^t \frac{e_{-w^k}(t, 0)}{(1 - w^2 \mu(w))e_{-w^k}(w, w - \rho(w))} \Delta w \right. \\
\times & \quad \left. \sum_{i, l \in P_N} \tilde{q}_{jil}(\beta_i^+ + \beta_l^\delta) \right] \bigg\| \psi - \hat{\psi} \bigg\| \quad (17)
\end{align*}
\]

By Property 1, (15), (16) and (17), there exist constants $w^{21}, w^{22} \in (0, w^k)$ such that

\[
\begin{align*}
\left\{ \begin{array}{l}
|u_i(t)|_{i \in N_N} \leq & \left[ e_{-w^{21}t} + \int_0^t \frac{e_{-w^{21}t}}{(1 - w^2 \mu)^T} e_{-w^k}(w - \rho(w)) \Delta w \right. \\
\times & \quad \sum_{j, k \in P_M} \tilde{p}_{ijk}(\beta_k^+ + \beta_j^\delta) \bigg\| \psi - \hat{\psi} \bigg\| \\
\leq & \left[ e_{-w^{21}t} + \frac{e_{w^{22}t}}{1 - w^2 \mu^T} \sum_{j, k \in P_M} \tilde{p}_{ijk}(\beta_k^+ + \beta_j^\delta) e_{-w^{h1}t} \right] \bigg\| \psi - \hat{\psi} \bigg\|,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
|v_j(t)|_{j \in N_M} \leq & \left[ e_{-w^{21}t} + \int_0^t \frac{e_{-w^{21}t}}{(1 - w^2 \mu^T)} e_{-w^k}(w - \rho(w)) \Delta w \right. \\
\times & \quad \sum_{i, l \in P_N} \tilde{q}_{jil}(\beta_i^+ + \beta_l^\delta) \bigg\| \psi - \hat{\psi} \bigg\| \\
\leq & \left[ e_{-w^{21}t} + \frac{e_{w^{22}t}}{1 - w^2 \mu^T} \sum_{i, l \in P_N} \tilde{q}_{jil}(\beta_i^+ + \beta_l^\delta) e_{-w^{h1}t} \right] \bigg\| \psi - \hat{\psi} \bigg\| \quad (18)
\end{align*}
\]

\]
and

\[ |u_i(t)|_{i \in P_N} \leq \ell \| \psi - \hat{\psi} \| e^{-w^{(1)}t}, \quad |v_j(t)|_{j \in P_M} \leq \ell \| \psi - \hat{\psi} \| e^{-w^{(2)}t}. \]  \hspace{1cm} (19)

Due to the basic fact \( \lim_{t \to +\infty} te^{-w^{(1)}t} = 0 \), from (18) and (19), we know that there exists a positive integer \( L \) such that

\[ |u_i(t)|_{i \in N} \leq \epsilon \| \psi - \hat{\psi} \|, \quad |v_j(t)|_{j \in M} \leq \epsilon \| \psi - \hat{\psi} \| \]

for all \( t > L \), where \( 0 < \epsilon < 1 \). Define a mapping \( \mathcal{G} : \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N, \mathcal{G}_M^N) \to \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N, \mathcal{G}_M^N) \) by \( \mathcal{G}(\psi) = (x_{\omega}(\psi), y_{\omega}(\psi)) \). By the invariance of \( \mathcal{G}_N^N \), we get \( \mathcal{G}(\psi) \in \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N, \mathcal{G}_M^N) \), i.e., \( \mathcal{G}(\mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N)) \subset \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N) \). It follows from \( \mathcal{G}^L(\psi) = \mathcal{G} \circ \mathcal{G} \circ \cdots \mathcal{G} = (x_{L\omega}(\psi), y_{L\omega}(\psi)) \) that

\[ \| \mathcal{G}^L(\psi) - \mathcal{G}^L(\hat{\psi}) \| = \max_{\theta \in \mathbb{T}_{[-t_D,0]}} \left\{ |u_i(L\omega + \theta; \psi)|, |v_j(L\omega + \theta; \psi)| \right\} \]

\[ \leq \epsilon \| \psi - \hat{\psi} \|. \]

This implies that \( \mathcal{G}^L \) is a contraction mapping. There exists a unique fixed point \( \tilde{\psi} \in \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N) \) such that \( \mathcal{G}^L(\tilde{\psi}) = \tilde{\psi} \) and hence \( \mathcal{G}^L(\tilde{\psi}) = \mathcal{G}(\mathcal{G}^L(\tilde{\psi})) = \mathcal{G}(\tilde{\psi}) = \tilde{\psi} \). So \( \mathcal{G}(\tilde{\psi}) = \tilde{\psi} \), i.e., \( (x_{\omega}(\tilde{\psi}), y_{\omega}(\tilde{\psi})) = \tilde{\psi} \). It is easy to check that \( (x(t, \tilde{\psi}), y(t, \tilde{\psi})) \) is a \( \omega \)-periodic solution of TLBAMs (1) in \( \mathcal{G}_M^N \). It follows from (18) and (19) that any solution with initial condition \( \psi \in \mathcal{S}(\mathbb{T}_{[-t_D,0]}; \mathcal{G}_N^N) \) converges to this \( \omega \)-periodic solution \( (x(t, \tilde{\psi}), y(t, \tilde{\psi})) \) exponentially as \( t \to +\infty \). The proof is complete. \( \blacksquare \)

Remark 1: There exist \( 2^n \) partitions of index set \( N \) and \( 2^m \) partitions of index set \( M \). Hence, there may exist \( 2^n + 2^m \) invariant sets \( \mathcal{G}_M^N \) satisfying with inequalities assumptions in Theorem 1 and the bound of \( \mathcal{G}_M^N \) is well estimated by \( (H_{P_N}) \) and \( (H_{P_M}) \). For special case \( \mathbb{T} = \mathbb{R} \), relative results on TLBAMs have been reported in [28].

Remark 2: For each invariant sets \( \mathcal{G}_M^N \), subsystem \((S_1)\) is independent, each neuron in index set \( P_N \) only interacts ones in \( P_M \). Subsystem \((S_2)\) is dependent on state evolution of subsystem \((S_1)\). To the authors’ knowledge, such dynamics of BAM on time scales have not been reported in the literature.

Corollary 1. Assume that \( a_i, b_j \in \mathbb{R}_{rd}^+ \) for all \( i \in N, j \in M \). If there exist constants \( \alpha_i^r, \beta_i^r \) with \( 0 < \alpha_i^r < \beta_i^r \) \( (r \in N) \) and \( \alpha_j^r, \beta_j^r \) with \( 0 < \alpha_j^r < \beta_j^r \)
\[(r \in \mathcal{M}) \text{ such that}\]

\[
(H^N) : \forall r \in \mathcal{N}, \quad \begin{cases}
-a_r(t)\alpha^+_r + \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{M}} p_{r,j,k}^-(t)\alpha^+_j \alpha^+_k \\
+ \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{M}} p_{r,j,k}^-(t)\beta^+_j \beta^+_k + I_r > 0, \\
-a_r(t)\beta^+_r + \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{M}} q_{r,j,k}^+(t)\alpha^-_j \alpha^-_k \\
+ \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{M}} q_{r,j,k}^+(t)\beta^-_j \beta^-_k + I_r < 0
\end{cases}
\]

and

\[
(H^M) : \forall r \in \mathcal{M}, \quad \begin{cases}
-b_r(t)\alpha^+_r + \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{N}} q_{r,j,k}^+(t)\alpha^-_j \alpha^-_k \\
+ \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{N}} q_{r,j,k}^+(t)\beta^-_j \beta^-_k + J_r > 0, \\
-b_r(t)\beta^+_r + \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{N}} q_{r,j,k}^+(t)\alpha^-_j \alpha^-_k \\
+ \sum_{j \in \mathcal{M}} \sum_{k \in \mathcal{N}} q_{r,j,k}^+(t)\beta^-_j \beta^-_k + J_r < 0
\end{cases}
\]

hold for \(\forall t \in \mathbb{T}\), then \(\mathcal{S}^N_M\) is invariant set of TLBAMs (1). Moreover, TLBAMs (1) has one periodic trajectory in \(\mathcal{S}^N_M\) and it exponentially attracts all trajectories of \(\mathcal{S}^M_M\), where \(\mathcal{S}^N_M = \mathcal{S}^N_X \times \mathcal{S}^M_Y\) and

\[
\mathcal{S}^N_X := (\alpha^+_1, \beta^+_1) \times \cdots \times (\alpha^+_n, \beta^+_n), \quad \mathcal{S}^M_Y := (\alpha^-_1, \beta^-_1) \times \cdots \times (\alpha^-_m, \beta^-_m).
\]

4. Illustrative Example

By using inequalities \((H^P_N), (H^N_N), (H^P_M)\) and \((H^N_M)\) in Theorem 1, we can find these invariant sets \(\mathcal{S}^N_M\) and obtain their periodic trajectories. By an example, we will emulate periodicity results on time scales for TLBAMs.

Example: Consider the following TLBAMs on time scale:

\[
\begin{align*}
x^\triangle(t) &= -a_1(t)x_1(t) + p_{111}(t)g(y_1(t-5))g(y_1(t-5)) + I_1(t) \\
x^\triangle(t) &= -a_2(t)x_2(t) + p_{111}(t)g(y_1(t-5))g(y_1(t-5)) + I_2(t) \\
y^\triangle(t) &= -b_1(t)y_1(t) + q_{111}(t)g(x_1(t))g(x_1(t)) \\
&\quad + q_{112}(t)g(x_2(t))g(x_2(t)) + q_{122}(t)g(x_2(t))g(x_2(t)) + I_1(t)
\end{align*}
\]

(20)

where \(a_1(t) = a_2(t) = b_1(t) = 0.8, p_{111}(t) = -5 + 0.01\sin\left(\frac{\pi}{4}t\right), p_{211}(t) = -5, q_{111}(t) = -3 + 0.01\cos\left(\frac{\pi}{4}t\right), q_{112}(t) = -3, q_{122}(t) = -1 + 0.01\sin\left(\frac{\pi}{4}t\right), I_1(t) = 4 + 2\cos\left(\frac{\pi}{4}t\right), I_2(t) = 3 + \sin\left(\frac{\pi}{4}t\right)\) and \(J_1(t) = 4 + 2\sin\left(\frac{\pi}{4}t\right)\). The time scale is
set to be \( T = \{ 8(k - 1) + t; t \in [0, 2] \cup [3, 4] \cup [5, 8], k \in \mathbb{N}_0 \} \). Then, there exist two cases for us to consider partitions of \( \mathcal{N} \).

**Case 1:** Taking \((P_N, N_N) = (\{1\}, \{2\})\) and \((P_M, N_M) = (\emptyset, \{1\})\), by Theorem 1, we have inequalities for invariant set as:

\[
\begin{cases}
-a_1(t)\alpha^+_1 + \bar{L}_1 > 0 \\
-a_1(t)\beta^+_1 + \bar{T}_1 > 0 \\
g_{111}(t)(\beta^+_1)^2 + g_{111}(t)(\alpha^+_1)^2 + \bar{T}_1 < 0
\end{cases}
\]

One can get a solution \( \alpha^+_1 = 2.4 \) and \( \beta^+_1 = 7.6 \) and

\[\mathcal{S}^N_M = \{ x | 2.4 < x_1 < 7.6, x_2 < 0 \} \times \{ y_1 | y_1 < 0 \}\]

is an invariant set. By Theorem 1, there exists a unique periodic trajectory in \( \mathcal{S}^N_M \) and exponentially attracts all the other trajectories of \( \mathcal{S}^N_M \).

**Case 2:** Taking \((P_N, N_N) = (\emptyset, \{1, 2\})\) and \((P_M, N_M) = (\{1\}, \emptyset)\), by Theorem 1, we have inequalities for invariant set as:

\[
\begin{cases}
p_{111}(t)(\beta^+_1)^2 + p_{111}(t)(\alpha^+_1)^2 + \bar{T}_1 < 0 \\
p_{211}(t)(\beta^+_1)^2 + p_{211}(t)(\alpha^+_1)^2 + \bar{T}_2 < 0 \\
-b_1(t)\alpha^+_1 + \bar{J}_1 > 0 \\
-b_1(t)\beta^+_1 + \bar{J}_1 < 0
\end{cases}
\]

One can get a solution \( \alpha^+_1 = 2.4 \) and \( \beta^+_1 = 8 \) and

\[\mathcal{S}^N_M = \{ x | x_1 < 0, x_2 < 0 \} \times \{ y_1 | 2.4 < y_1 < 8 \}\]

is an invariant set. By Theorem 1, there exists a unique periodic trajectory in \( \mathcal{S}^N_M \) and exponentially attracts all the other trajectories of \( \mathcal{S}^N_M \).

**Remark 3.** As \((P_M, N_M) = (\emptyset, \{1\})\), \((P_N, N_N)\) can be \((\{2\}, \{1\})\), \((\{1, 2\}, \emptyset)\) and \((\emptyset, \{1, 2\})\); As \((P_M, N_M) = (\{1\}, \emptyset)\), \((P_N, N_N)\) can be \((\{2\}, \{1\})\), \((\{1\}, \{2\})\) and \((\{1, 2\}, \emptyset)\). However, we have not found any solutions satisfying \((H^{P_M})\), \((H^{N_M})\), \((H^{P_N})\) and \((H^{N_N})\) in Theorem 1.

### 5. Conclusion

The main contribution of this paper is the analysis of invariant sets and periodicity for MBA networks with PLT functions on time scales. Based on properties of exponential functions, new conditions for invariance and globally exponentially attractivity of periodic solutions of BAM networks have been established. The approach to periodicity of BAM on time scales is different from ones in the literature and can be regarded as a complement to the existing ones [21-24]. To the end, an example is employed to illustrate the theories.

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