Approximations of objective function in bi-criteria optimization problems

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Abstract
In this paper we study approximation methods for solving bi-criteria optimization problems. Initial problem is approximated by a new one which has the same constraints, while the components of the objective are replaced by their approximation functions. Conditions such that efficient solution of the approximate problem will remain efficient for initial problem and reciprocally are studied. Numerical examples are developed to emphasize the importance of these conditions.

Keywords: efficient solution, bi-criteria optimization, η-approximation, invex and incave function

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1. Introduction

Real-world and theoretical situations are often generating optimization problems. Depending on the type of mathematical model associated to the analyzed phenomena, different techniques might be used to compute an optimal solution, if it exists.

Optimization problems are solved either by analytical methods (compute the exact solution based on mathematical proofs) or by numerical methods (approximate the solution using appropriate iterations).

"Scalarization" methods [6] (weighting problem, kth objective Lagrangian problem, kth objective ε - constrained problem) are often used for solving bi-criteria optimization problems. They consists of transforming the bi-criteria optimization problem into two inter- correlated optimization problems with restrictions or transforming the bi-criteria optimization problem into an equivalent parametric optimization problem. The equivalent problems are solved, providing an optimal solution (if exists), which generates also the efficient solution for the

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bi-criteria optimization problem. In trade-off methods the importance degree of each component of the objective function might be established a priori or during the process. [16, 21, 24] illustrate contributions to the development of trade-off methods.

In case of multi-objective optimization problems, the numerical methods might be divided [9] into algorithms inspired from biology [10, 17, 18, 19], from physics [15, 20], from geography [13, 14] or from social culture [8, 25].

This article is analyzing conditions such that efficient solution of a certain approximate problem will remain efficient for the initial problem and reciprocally. Antczak [2, 3, 4], Duca [5, 7, 11, 12, 22], Popovici [1, 23] have contributed, among others, to this method of solving optimization problems.

Novelty of our work consists in developing conditions such that the efficient solution of the initial optimization problem will remain efficient for an approximate problem and reciprocally, where the approximate problem consists of approximate functions for the components of the objective and the same feasible set.

Roots of our research in approximation theory is related to our work for shaving the peak load of energy production. Mathematical models used for peak load shaving might be highly complex. A solving approach based on approximated problems might be more efficient compared with trade-off methods.

2. Basic concepts

Let $X$ be a set in $\mathbb{R}^n$, $x_0$ an interior point of $X$, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$. If $f$ is differentiable at $x_0$ then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0) \eta(x, x_0)$$

and call it first $\eta$–approximation of $f$

and if $f$ is twice differentiable at $x_0$ then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0) \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 f(x_0) \eta(x, x_0).$$

and call it second $\eta$–approximation of $f$.

**Definition 2.1.** Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $f : X \to \mathbb{R}$ a function differentiable at $x_0$ and $\eta : X \times X \to X$. Then function $f$ is:

**invex** at $x_0$ with respect to $\eta$ if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0) \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

**inconvex** at $x_0$ with respect to $\eta$ if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0) \eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$
If function $f$ is invex, respectively incave we denote invex$^1$, respectively incave$^1$.

**Definition 2.2.** Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $f : X \to \mathbb{R}$ a function twice differentiable at $x_0$ and $\eta : X \times X \to X$. Then function $f$ is:

**second order invex** at $x_0$ with respect to $\eta$ if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0) \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 f(x_0) \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

**second order incave** at $x_0$ with respect to $\eta$ if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0) \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 f(x_0) \eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

If function $f$ is second order invex, respectively second order incave we denote invex$^2$, respectively incave$^2$.

Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X, \eta : X \times X \to X$, $T$ and $S$ index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t, h_s : X \to \mathbb{R}, (t \in T, s \in S)$ functions.

We consider the bi-criteria optimization problem $\left( P_0^{0,0} \right)$, defined as:

$$\begin{cases} 
\min (f_1, f_2)(x) \\
x = (x_1, x_2, ..., x_n) \in X \\
g_t(x) \leq 0, \ t \in T \\
h_s(x) = 0, \ s \in S.
\end{cases}$$

Assuming that functions $f_1, f_2$ are differentiable of order $i, j \in \{1, 2\}$, we will approximate original problem $\left( P_0^{0,0} \right)$ by problems $\left( P_0^{i,j} \right)$:

$$\begin{cases} 
\min \left( F_1^i, F_2^j \right)(x) \\
x = (x_1, x_2, ..., x_n) \in X \\
g_t(x) \leq 0, \ t \in T \\
h_s(x) = 0, \ s \in S
\end{cases}$$

where $(i, j) \in \{(1,0), (1,1), (2,0), (2,1), (2,2)\}$, and $F_1^0 = f_1, F_2^0 = f_2$.

We denote by

$$\mathcal{F}^0 = \{x \in X : g_t(x) \leq 0, \ t \in T, h_s(x) = 0, \ s \in S\}$$

the set of feasible solutions for bi-criteria optimization problem $\left( P_0^{i,j} \right)$, where $(i, j) \in \{(1,0), (1,1), (2,0), (2,1), (2,2)\}$. 
3. Approximate problems and relation to initial problem

In this section we will study the conditions such that efficient solution of approximated problems \((P^{1,0}_0), (P^{1,1}_0), (P^{2,0}_0), (P^{2,1}_0)\) and \((P^{2,2}_0)\) will remain efficient also for original problem \((P^{0,0}_0)\) and reciprocally.

**Theorem 3.1.** Let \(X\) be a nonempty set of \(\mathbb{R}^n\), \(x_0\) an interior point of \(X\), \(\eta : X \times X \to X\), \(T\) and \(S\) index sets, \(f = (f_1, f_2) : X \to \mathbb{R}^2\) and \(g_t, h_s : X \to \mathbb{R}, (t \in T, s \in S)\) functions.

Assume that:

a) \(f_1\) is differentiable at \(x_0\) and invex\(^1\) at \(x_0\) with respect to \(\eta\).

b) \(\eta(x_0, x_0) = 0\).

If \(x_0\) is an efficient solution for \((P^{1,0}_0)\), then \(x_0\) is an efficient solution for \((P^{0,0}_0)\).

**Proof.** \(x_0\) being an efficient solution for \((P^{1,0}_0)\) implies that

\[
\exists \bar{x} \in \mathcal{F}^0 \text{ s.t. } (F_1^1(\bar{x}), f_2(\bar{x})) \leq (F_1^1(x_0), f_2(x_0)).
\]

Let’s assume that \(x_0\) is not an efficient solution for \((P^{0,0}_0)\). It means

\[
\exists y \in \mathcal{F}^0 \text{ s.t. } (f_1(y), f_2(y)) \leq (f_1(x_0), f_2(x_0))
\]

which implies that \(\exists y \in \mathcal{F}^0 \text{ s.t.}\)

\[
\begin{cases}
  f_1(y) < f_1(x_0) \\
  f_2(y) \leq f_2(x_0)
\end{cases}
\]

(1)

or

\[
\begin{cases}
  f_1(y) \leq f_1(x_0) \\
  f_2(y) < f_2(x_0)
\end{cases}
\]

(2)

Because \(f_1\) is invex\(^1\) at \(x_0\) with respect to \(\eta\) we get \(F_1^1(y) \leq f_1(y), \forall y \in \mathcal{F}^0\).
Because \(\eta(x_0, x_0) = 0\) we get \(f_1(x_0) = F_1^1(x_0)\).
Thus from (1) we get that \(\exists y \in \mathcal{F}^0 \text{ s.t.}\)

\[
\begin{cases}
  F_1^1(y) < F_1^1(x_0) \\
  f_2(y) \leq f_2(x_0)
\end{cases}
\]

which contradicts the efficiency of \(x_0\) for \((P^{1,0}_0)\) and from (2) we get that \(\exists y \in \mathcal{F}^0 \text{ s.t.}\)

\[
\begin{cases}
  F_1^1(y) \leq F_1^1(x_0) \\
  f_2(y) < f_2(x_0)
\end{cases}
\]
which contradicts the efficiency of $x_0$ for $\left(P_0^{1,0}\right)$.

In conclusion $x_0$ is an efficient solution for $\left(P_0^{0,0}\right)$. ■

**Theorem 3.2.** Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $\eta : X \times X \to X$, $T$ and $S$ index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t, h_s : X \to \mathbb{R}, (t \in T, s \in S)$ functions.

Assume that:

a) $f_1$ is differentiable at $x_0$ and incave\(^1\) at $x_0$ with respect to $\eta$.

b) $\eta(x_0, x_0) = 0$.

If $x_0$ is an efficient solution for $\left(P_0^{0,0}\right)$, then $x_0$ is an efficient solution for $\left(P_0^{1,0}\right)$.

**Proof.** $x_0$ being an efficient solution for $\left(P_0^{0,0}\right)$ implies that

$$\not\exists x \in \mathcal{F}^0 \text{ s.t. } (f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)).$$

Let's assume that $x_0$ is not an efficient solution for $\left(P_0^{1,0}\right)$. It means

$$\exists y \in \mathcal{F}^0 \text{ s.t. } (F_1^1(y), f_2(y)) \leq (F_1^1(x_0), f_2(x_0))$$

which implies that $\exists y \in \mathcal{F}^0 \text{ s.t.}$

$$\begin{cases} F_1^1(y) < F_1^1(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases} \quad (3)$$

or

$$\begin{cases} F_1^1(y) \leq F_1^1(x_0) \\ f_2(y) < f_2(x_0) \end{cases} \quad (4)$$

Because $f_1$ is incave\(^1\) at $x_0$ with respect to $\eta$ we get $f_1(y) \leq F_1^1(y), \forall y \in \mathcal{F}^0$.

Because $\eta(x_0, x_0) = 0$ we get $f_1(x_0) = F_1^1(x_0)$. Thus from (3) we get that $\exists y \in \mathcal{F}^0 \text{ s.t.}$

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leq f_2(x_0) \end{cases}$$

which contradicts the efficiency of $x_0$ for $\left(P_0^{0,0}\right)$ and from (4) we get that $\exists y \in \mathcal{F}^0 \text{ s.t.}$

$$\begin{cases} f_1(y) \leq f_1(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts the efficiency of $x_0$ for $\left(P_0^{0,0}\right)$.

In conclusion $x_0$ is an efficient solution for $\left(P_0^{1,0}\right)$. ■
Example 3.1 (diff. solutions for initial and approximate problems). .
If condition a) from Theorem 3.2 is not satisfied, it might be possible either
that initial and approximate problems have the same solution or different. The
following example is presenting the case when the two problems have different
solutions.
Let the initial bi-criteria optimization problem \( (P_{0}^{0,0}) \) be:

\[
\begin{align*}
\text{min } f(x) &= (x_1^2 + x_2^2; x_1 - 2x_2) \\
-x_1 - x_2 + 2 &\leq 0 \\
x_1; x_2 &\geq 0
\end{align*}
\]

\( x^0 = (1, 1) \) is an efficient solution for problem (5) and the value of function \( f \)
is \( f(1, 1) = (2, -1) \).
To compute the approximate problem \( (P_{0}^{1,0}) \) we have to calculate the first \( \eta \)-
approximation of \( f_1 \). Thus

\[
F_1^1(x) = f_1(x^0) + \nabla f(x^0) \eta(x, x^0).
\]

Considering \( \eta(x, x^0) = x - x^0 \), we obtain

\[
F_1^1(x) = 2x_1 + 2x_2 - 2.
\]

and the approximate problem \( (P_{0}^{1,0}) \) will be

\[
\begin{align*}
\text{min } F(x) &= (2x_1 + 2x_2 - 2; x_1 - 2x_2) \\
-x_1 - x_2 + 2 &\leq 0 \\
x_1; x_2 &\geq 0
\end{align*}
\]

Thus

\[
F(x^0) = F(1, 1) = (2, -1) > (2, -4) = F(0, 2),
\]

which proves that \( x^0 = (1, 1) \) is not an efficient solution for problem \( (P_{0}^{1,0}) \). In
conclusion, efficient solution of problem (5) doesn’t remain efficient also for
approximate problem (6), proving that incavity of \( f_1 \) from condition a) of Theorem
3.2 is essential.

Remark 3.1. Conditions such that efficient solution of problem \( (P_{0}^{1,1}) \) will
remain efficient for problem \( (P_{0}^{0,0}) \) and reciprocally have been studied in [12].

Theorem 3.3. Let \( X \) be a nonempty set of \( \mathbb{R}^n \), \( x_0 \) an interior point of \( X \), \( \eta : X \times X \rightarrow X \), \( T \) and \( S \) index sets, \( f = (f_1, f_2) : X \rightarrow \mathbb{R}^2 \) and \( g_t, h_s : X \rightarrow \mathbb{R}, (t \in T, s \in S) \) functions.
Assume that:

a) \( f_1 \) is twice differentiable at \( x_0 \) and \( \text{inv}^2 \) at \( x_0 \) with respect to \( \eta \).
b) \( \eta(x_0, x_0) = 0 \).

If \( x_0 \) is an efficient solution for \( (P_{0}^{2,0}) \), then \( x_0 \) is an efficient solution for \( (P_{0}^{0,0}) \).

**Proof.** Proof is similar with Theorem 3.1. ■

**Theorem 3.4.** Let \( X \) be a nonempty set of \( \mathbb{R}^n \), \( x_0 \) an interior point of \( X \), \( \eta : X \times X \rightarrow X, T \) and \( S \) index sets, \( f = (f_1, f_2) : X \rightarrow \mathbb{R}^2 \) and \( g_t, h_s : X \rightarrow \mathbb{R}, (t \in T, s \in S) \) functions.

Assume that:

a) \( f_1 \) is twice differentiable at \( x_0 \) and incave\(^2\) at \( x_0 \) with respect to \( \eta \).
b) \( \eta(x_0, x_0) = 0 \).

If \( x_0 \) is an efficient solution for \( (P_{0}^{0,0}) \), then \( x_0 \) is an efficient solution for \( (P_{0}^{2,0}) \).

**Proof.** Proof is similar with Theorem 3.2. ■

**Example 3.2** (same solution for initial and approximate problems).

Let the initial bi-criteria optimisation problem \( (P_{0}^{0,0}) \) be:

\[
\begin{align*}
\min & \quad (x_1^2 + x_1 x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2) \\
& \quad -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\
& \quad 4x_1 + x_2 - 20 \leq 0 \\
& \quad x_1; x_2 \geq 0 \\
\end{align*}
\]

\( (7) \)

An efficient solution of problem \( (7) \) is \( x^0 = (3, 8) \). Starting from it, the following approximation in \( x^0 \) for first component of objective function has to be computed

\[
F_1^2(x) = f_1(x^0) + \nabla f_1(x^0) \eta(x, x^0) + \frac{1}{2} \eta(x, x^0)^T \nabla^2 f_1(x^0) \eta(x, x^0).
\]

Considering \( \eta(x, x^0) = x - x^0 \) we obtain

\[
F_1^2(x) = x_1^2 + x_1 x_2 + x_2^2 - 19.25x_1 - 19.875x_2
\]

and the corresponding approximate problem \( (P_{0}^{2,0}) \) is:

\[
\begin{align*}
\min & \quad (x_1^2 + x_1 x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2) \\
& \quad -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\
& \quad 4x_1 + x_2 - 20 \leq 0 \\
& \quad x_1; x_2 \geq 0 \\
\end{align*}
\]

\( (8) \)

which is identical with initial problem \( (7) \) and thus they have the same efficient solution.
Remark 3.2. Example 3.2 shows that if second order incavity of $f_1$ from condition a) of Theorem 3.4 is not satisfied it might be possible to obtain the same efficient solution.

Theorem 3.5. Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $\eta : X \times X \rightarrow X$, $T$ and $S$ index sets, $f = (f_1, f_2) : X \rightarrow \mathbb{R}^2$ and $g_k, h_s : X \rightarrow \mathbb{R}, (t \in T, s \in S)$ functions.

Assume that:

a) $f_1$ is twice differentiable at $x_0$ and incave$^2$ at $x_0$ with respect to $\eta$.

b) $f_2$ is differentiable at $x_0$ and incave$^1$ at $x_0$ with respect to $\eta$.

c) $\eta(x_0, x_0) = 0$.

If $x_0$ is an efficient solution for $(P_{0}^{2,1})$, then $x_0$ is an efficient solution for $(P_{0}^{0,0})$.

Proof. Proof is similar with Theorem 3.1. ■

Theorem 3.6. Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $\eta : X \times X \rightarrow X$, $T$ and $S$ index sets, $f = (f_1, f_2) : X \rightarrow \mathbb{R}^2$ and $g_k, h_s : X \rightarrow \mathbb{R}, (t \in T, s \in S)$ functions.

Assume that:

a) $f_1$ is twice differentiable at $x_0$ and incave$^2$ at $x_0$ with respect to $\eta$.

b) $f_2$ is differentiable at $x_0$ and incave$^1$ at $x_0$ with respect to $\eta$.

c) $\eta(x_0, x_0) = 0$.

If $x_0$ is an efficient solution for $(P_{0}^{0,0})$, then $x_0$ is an efficient solution for $(P_{0}^{2,1})$.

Proof. Proof is similar with Theorem 3.2. ■

Theorem 3.7. Let $X$ be a nonempty set of $\mathbb{R}^n$, $x_0$ an interior point of $X$, $\eta : X \times X \rightarrow X$, $T$ and $S$ index sets, $f = (f_1, f_2) : X \rightarrow \mathbb{R}^2$ and $g_k, h_s : X \rightarrow \mathbb{R}, (t \in T, s \in S)$ functions.

Assume that:

a) $f_1$ is twice differentiable at $x_0$ and incave$^2$ at $x_0$ with respect to $\eta$.

b) $f_2$ is twice differentiable at $x_0$ and incave$^2$ at $x_0$ with respect to $\eta$.

c) $\eta(x_0, x_0) = 0$. 
If \( x_0 \) is an efficient solution for \((P_{0}^{2,2})\), then \( x_0 \) is an efficient solution for \((P_{0}^{0,0})\).

**Proof.** Proof is similar with Theorem 3.1. ■

**Theorem 3.8.** Let \( X \) be a nonempty set of \( \mathbb{R}^n \), \( x_0 \) an interior point of \( X \), \( \eta : X \times X \rightarrow X \), \( T \) and \( S \) index sets, \( f = (f_1, f_2) : X \rightarrow \mathbb{R}^2 \) and \( g_t, h_s : X \rightarrow \mathbb{R} \) \((t \in T, s \in S)\) functions.

Assume that:

a) \( f_1 \) is twice differentiable at \( x_0 \) and incave\(^2\) at \( x_0 \) with respect to \( \eta \).

b) \( f_2 \) is twice differentiable at \( x_0 \) and incave\(^2\) at \( x_0 \) with respect to \( \eta \).

c) \( \eta(x_0, x_0) = 0 \).

If \( x_0 \) is an efficient solution for \((P_{0}^{0,0})\), then \( x_0 \) is an efficient solution for \((P_{0}^{2,2})\).

**Proof.** Proof is similar with Theorem 3.2. ■

4. References


