Z -Transform Inversion of The Gamma Function

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Abstract
In this work, we derive the mathematical expression for the inverse Z*-transform of the gamma function* $\Gamma(z)$ *where* n *represents a nonnegative integer and* z *is a complex variable. The corresponding gamma sequence, denoted by* $\gamma(n)$, *is found by the method of complex residues and numerical values are given for* $n=0$ *to* $n=21$. *Background basic theoretical material including a few* Z*-transform companion examples is provided to make our exposition self complete.*

Keywords: complex residues, gamma function, real sequences, special functions, Z-transform.

1. Introduction

The Z-transform, known also as the Hurewicz transform, is a discrete transform version derived from the Laplace or L-transform as described in [1]-[4]. In the same way as the L-transform is applied to the analysis and synthesis of *continuous linear systems*, the use of the Z-transform appears frequently in the analysis and design of *discrete linear systems*, treated in extension, for example in [5] to [7]. Another name for the background material briefly discussed here is known as the theory of *generating functions*, commonly used in other mathematical disciplines such as probability theory and combinatorial analysis, for which [8] and [9] are excellent expositions of this approach. In this paper, we are mainly concerned with the unilateral or negative Z-transform and will refer to it simply as the Z-transform. Detailed mathematical presentations relative to the Z-transform and related transforms can be perused in references [10] to [13].

The present work is organized as follows: Section 2 provides some mathematical preliminaries about the Z-transform with the purpose of establishing the conceptual framework and the appropriate symbolism for the derivation given in Section 3, in which we explain with sufficient detail the mathematical steps required to obtain the gamma sequence. In Section 4 we compute a specific range of values of the gamma sequence and discuss the general pattern of its terms. Finally, the conclusion about the solution exposed here is given in Section 5.

2. Direct and Inverse Z-Transforms

Let $\varphi:[0,\infty)\to\mathbb{R}$ be a continuous real valued function. Then, a *sampled* or *discrete* function, denoted by $\tilde{\varphi}:\{nT \mid n\in\mathbb{N}, T\in\mathbb{R}^+\}$ where T represents the *sampling period*, can be associated in a natural way to φ by means of the following expression,

$$\tilde{\varphi}(t)=\varphi(t)\sum_{n=0}^{\infty}\delta(t-nT)=\sum_{n=0}^{\infty}\varphi(nT)\delta(t-nT). \quad (1)$$

In (1), δ represents Dirac's unitary impulse function. If the scale t/T is used instead of t, then the domain of $\tilde{\varphi}$ equals the set of natural numbers $\mathbb{N}$. Therefore, we can assign a *real sequence* of numbers equivalent to $\tilde{\varphi}$. If φ and ψ are two continuous functions defined on $[0,\infty)$ such that $\varphi(t)\neq\psi(t)$ for all $t\geq 0$, it may happen that $\tilde{\varphi}(t)=\tilde{\psi}(t)$ for some values of t, e. g., take $\varphi(t)=1$ and $\psi(t)=\cos(2\pi t/T)$; in such case, to distinguish between the sequences associated to $\tilde{\varphi}$ and $\tilde{\psi}$, the period T must be lowered to produce a finer sampling. Applying the Laplace transform to the sampled function $\tilde{\varphi}$ we have that,

$$\mathrm{L}\{\tilde{\varphi}(t)\}=\int_0^{\infty}\tilde{\varphi}(t)e^{-st}dt=\tilde{\Phi}(s)=\sum_{n=0}^{\infty}\varphi(nT)\,\mathrm{L}\{\delta(t-nT)\}=\sum_{n=0}^{\infty}\varphi(nT)e^{-nTs}. \quad (2)$$

Since, $\tilde{\Phi}(s)=\tilde{\Phi}(s+2\pi ni)$ with $\mathbb{N}$ and $i=\sqrt{-1}$, then $\tilde{\Phi}$ is a periodic function of s. Making the change of variable, $s=T^{-1}\ln z$ or equivalently $z=e^{sT}$, we obtain the formal expression that defines the (unilateral or negative) *Z- transform*. In symbols,

$$\mathrm{Z}\{\varphi(n)\}=\tilde{\Phi}(s)\Big|_{s=T^{-1}\ln z}=\sum_{n=0}^{\infty}\varphi(nT)z^{-n}. \quad (3)$$

It is customary to use the following notation:

$$\mathrm{Z}\{\varphi\}=\mathrm{Z}\{\varphi(n)\}=\Phi(z)=\sum_{n=0}^{\infty}\varphi(n)z^{-n}, \quad (4)$$

where the period T has been dropped out since it is a fixed known value (or else we can take $T=1$). The correspondence between φ and Φ under Z will be referred as a Z-*transform pair* or simply a transform pair and symbolized by $\varphi(n)\leftrightarrow\Phi(z)$. Notice that the Z-transform is a *complex power series* whose coefficients are the sampled values of the transformed function. If $s=\sigma+i\omega$, where, then its modulus is given by, $|z|=|e^{sT}|=|e^{\sigma T}||e^{i\omega T}|=|e^{\sigma T}|$. From Fig. 1, which depicts the complex mapping $z=e^{sT}$ from to itself, the geometrical meaning of this result can be explained as follows. $\sigma,\omega\in\mathbb{R}$ Similarly, if $\sigma>0$ then $|z|>1$, i. e., the right complex half-plane goes to the exterior of the complex unit circle. In the case that, $\sigma=0$ then $|z|=1$, thus the imaginary axis $i\omega$

maps to the circumference of the complex unit circle. In other words, Fig. 1 shows the relation between the *stability regions* specified, respectively, by the L and Z transforms. Furthermore, the notion of stability of a discrete linear system is closely related to the *region of convergence* (ROC) of the power series $\Phi(z)$ given by (4).

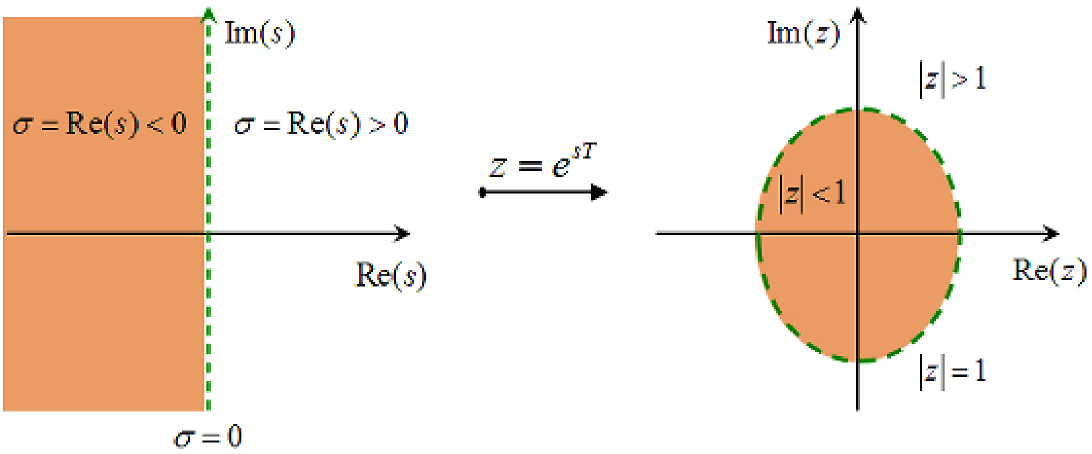

Fig. 1. Graphical representation of the complex map $z = \exp(sT)$ from $\mathbb{C}$ to itself. Left: stability regions (shaded and dashed) in the complex plane for the Laplace transform. Right: corresponding stability regions (shaded and dashed) for the Z-transform.

Associated to the "direct" Z-transform, the *inverse Z-transform* gives a discrete function $\varphi(n)$ for $n \geq 0$. Its formula is given by the *complex contour integral*,

$$\varphi(n) = \mathrm{Z}^{-1}\{\Phi(z)\} = \frac{1}{2\pi i}\oint_C \Phi(z) z^{n-1} dz, \tag{5}$$

where C is a circular path in the complex plane enclosing all singularities of $\Phi(z)z^{n-1}$, that is, all points z such that $\Phi(z)z^{n-1} \to \infty$. The equality given by (5) can be obtained in the following way. The Laplace transform inversion formula establishes that,

$$\varphi(t) = \mathrm{L}^{-1}\{\Phi(s)\} = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{\sigma-i\omega}^{\sigma+i\omega} \Phi(s)e^{st} ds. \tag{6}$$

Then setting $t = n$ and changing e^s by z, i. e., making $s = \ln z$, we have

$$\varphi(n) = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{\sigma-i\omega}^{\sigma+i\omega} \Phi(s)e^{sn} ds = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{e^{\sigma-i\omega}}^{e^{\sigma+i\omega}} \Phi(\ln z) z^n d(\ln z), \tag{7}$$

from which (5) follows.

We will take into consideration the following elementary sequences. The *unitary impulse* or Dirac sequence defined by $\delta(n) = 1$ for $n = 0$ and $\delta(n) = 0$ for $n \neq 0$, and the *unitary step* or Heaviside sequence defined by, $u(n) = 1$ for $n \geq 0$ and $u(n) = 0$ for $n < 0$. In similar fashion, the unitary impulse and the unitary step sequences *shifted* or *translated* at $m > 0$ are specified as $\delta(n-m) = 1$ for $n = m$ and $\delta(n-m) = 0$ for $n \neq m$, and $u(n-m) = 1$ for $n \geq m$ and $u(n-m) = 0$ for $n < m$. It is not difficult to verify that the corresponding Z-transforms are given, respectively by, $Z\{\delta(n)\}=1$, $Z\{u(n)\}=z/(z-1)$ for $|z|>1$, $Z\{\delta(n-m)\}=z^{-m}$ for $|z|>0$, and $Z\{u(n-m)\}=z^{1-m}/(z-1)$ for $|z|>1$ and $z\neq 0$.

3. Inversion by Complex Residues

For some applications such as the solution of difference equations it is mandatory to find a sequence φ given its Z-transform Φ. In other words, given the Z-transform as a function of z, i.e., $\Phi(z)$, it is required to calculate the corresponding sequence of n, $\varphi(n)$, such that $Z\{\varphi(n)\}=\Phi(z)$. This type of problem is commonly known as *inversion* or *inverse transformation* for which partial fraction development, complex residues determination, and power series expansions are three ways of accomplishing it, then, in simplified form we can write $\varphi(n)=Z^{-1}\{\Phi(z)\}$. For example, inversion of $\Phi(z)$ using complex residues follows immediately from the definition of the inverse Z-transform given in (5). Thus, recalling that Cauchy's residue theorem establishes the known relationship given by

$$\oint_C \Phi(z)z^{n-1}dz = 2\pi i\sum_{k=1}^{m}\text{res}_{z=z_k}[\Phi(z)z^{n-1}], \tag{8}$$

we can calculate φ by evaluating the following summation,

$$\varphi(n)=\sum_{k=1}^{m}\text{res}_{z=z_k}[\Phi(z)z^{n-1}], \tag{9}$$

where each z_k is a pole of $\Phi(z)z^{n-1}$ for $k\in\{1,\ldots,m\}$. As an example, let us find the inverse Z-transform of $\Phi(z)=e^{1/z}$. In this particular case, we find the sequence by evaluating the complex integral and applying Cauchy's integral theorem to each term. The following chain of equalities accomplishes our purpose,

$$\begin{aligned}\varphi(n)&=\frac{1}{2\pi i}\oint_C e^{z^{-1}}z^{n-1}dz=\frac{1}{2\pi i}\oint_C\Big(\sum_{m=0}^{\infty}\frac{z^{-m}}{m!}\Big)z^{n-1}dz\\&=\frac{1}{2\pi i}\oint_C\sum_{m=0}^{\infty}\frac{z^{n-m-1}}{m!}dz=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{m!}\oint_C z^{n-m-1}dz\\&=\frac{1}{2\pi i}\Big[\sum_{m=0}^{n-1}\frac{1}{m!}\oint_C z^{n-m-1}dz+\frac{1}{n!}\oint_C\frac{dz}{z}+\sum_{m=n+1}^{\infty}\frac{1}{m!}\oint_C z^{n-m-1}dz\Big]\\&=\frac{1}{2\pi i}\Big(0+\frac{2\pi i}{n!}+0\Big)=\frac{1}{n!}.\end{aligned}$$

Since the sequence obtained is valid for $n\geq 0$ we obtain $\varphi(n)=u(n)/n!$. As an extension of the previous example we can find the inverse Z-transform of a function Φ that is *analytic* or *holomorphic* in the region $|z|>0$ (the complex plane except the origin). By hypothesis, $\Phi(z)$ can be expressed as a Laurent power series about $z=0$, thus,

$$\begin{aligned}\Phi(z)&=\sum_{m=-\infty}^{\infty}a_m z^m=\sum_{m=-\infty}^{0}a_m z^m+\sum_{m=1}^{\infty}a_m z^m\\&=\sum_{n=0}^{\infty}a_{-n}z^{-n}+\sum_{m=1}^{\infty}a_m z^m=Z\{\varphi(n)\}+\sum_{m=1}^{\infty}a_m z^m.\end{aligned}$$

Restricting the class of analytic functions to those whose infinite non-constant polynomial part is zero, we have that $a_m = 0$ for $m \geq 1$. In that case, $\varphi(n) = \mathrm{Z}^{-1}\{\Phi(z)\} = a_{-n}$ for any $n \in \mathbb{N}$. On the other hand, recall that the m th-coefficient in the Laurent series is given by,

$$a_m = \frac{1}{2\pi i}\oint_C \frac{\Phi(\zeta)}{\zeta^{m+1}} d\zeta.$$

If we let $m = -n$, the corresponding integral, except for the name of the integration variable, equals $\varphi(n)$. Therefore, we have that,

$$\varphi(n) = a_{-n} = \frac{1}{2\pi i}\oint_C \frac{\Phi(\zeta)}{\zeta^{-n+1}} d\zeta = \frac{1}{2\pi i}\oint_C \Phi(\zeta)\zeta^{n-1} d\zeta,$$

is the same inversion formula introduced earlier in (5). Furthermore, under the given conditions imposed on Φ, it turns out that the previous explanation is another way of defining the inverse transform Z^{-1} of Φ *without* recurring to the Laplace transform. An alternative argument based on Cauchy's integral theorem provides us with the same result. Specifically,

$$\varphi(n) = \frac{1}{2\pi i}\oint_C \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_C \Big(\sum_{m=-\infty}^{\infty} a_m z^m\Big) z^{n-1} dz$$
$$= \sum_{m=-\infty}^{\infty} \frac{a_m}{2\pi i}\oint_C z^{m+n-1} dz = \sum_{m \neq -n}^{\infty} \frac{a_m}{2\pi i}\oint_C z^{m+n-1} dz + \frac{a_{-n}}{2\pi i}\oint_C \frac{dz}{z} = a_{-n}.$$

4. Z-Transform Inversion of the Gamma Function

We assume the reader is acquainted with the basic theory behind the gamma special function, otherwise references [14] and [15] provide comprehensive foundations or [16] gives a modern and recent exposition. A none trivial problem is posed by the inverse Z-transform of the gamma function denoted by $\Gamma(z)$. In other words, find the sequence given by, $\gamma(n) = \mathrm{Z}^{-1}\{\Gamma(z)\}$ for $n \geq 0$. Due to the nature of $\Gamma(z)$ as a *special function* our development here is limited to the application of the complex residues method for finding inverse transforms as briefly discussed in Section 3. In order to make clear some onward calculations, we first determine the value of the following integral,

$$\oint_C \Gamma(z) dz,$$

where, C is the contour shown in Fig. 2, enclosing all *singularities* or *poles* (all of first order) of $\Gamma(z)$ given by the elements of the set $\mathbb{Z}_0^- = \{z \in \mathbb{Z} : z \leq 0\}$. Instead of taking for $\Gamma(z)$ its integral representation we use the alternative definition contributed by Gauss, i. e., $\Gamma(z) = \lim_{m\to\infty} \Gamma_m(z)$ where,

$$\Gamma_m(z) = \frac{m!m^z}{z(z+1)\cdots(z+m)} = \frac{m!m^z}{\prod_{j=0}^{m}(z+j)}.$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \Gamma(z) dz = \frac{1}{2\pi i}\oint_C \lim_{m\to\infty} \Gamma_m(z) dz = \sum_{k=0}^{\infty} \mathrm{res}_{z=-k}[\lim_{m\to\infty} \Gamma_m(z)]. \tag{10}$$

Note that the sum of residues is an *infinite series* whose general coefficient is found by evaluating the limit of $\Gamma_m(-k)$ when $m \to \infty$ and $k \in \mathbb{Z}_0^-$.

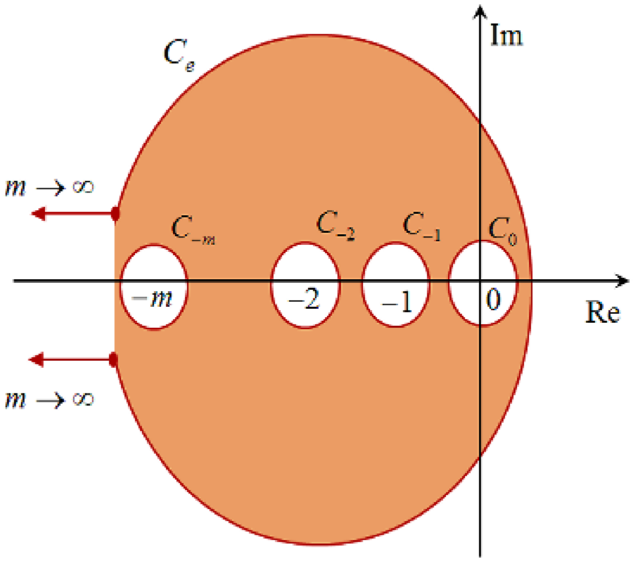

Fig. 2. The contour C in the complex plane $\mathbb{C}$ enclosing (counter clockwise) all simple poles of the gamma function forms a multiply connected region. This contour is made up of the external contour C_e and the internal contours $C_0, C_{-1}, C_{-2}, \ldots, C_{-m}$, hence, $\mathrm{C} = C_e \cup C_0 \cup \ldots \cup C_{-m}$. If $m \to \infty$ the radius of C_e gets bigger in order to contain the internal contours of more singularities as represented by the red arrows pointing to the left.

The details are shown next,

$$\begin{aligned}
\operatorname{res}_{z=-k}[\lim_{m\to\infty}\Gamma_m(z)] &= \lim_{m\to\infty}[(z+k)m!m^z / \prod_{j=0}^{m}(z+j)\big|_{z=-k}] = \lim_{m\to\infty}[m!m^z / \prod_{j=0}^{k-1}(z+j)\prod_{j=k+1}^{m}(z+j)\big|_{z=-k}] \\
&= \lim_{m\to\infty} m!m^{-k} / \prod_{j=0}^{k-1}(-1)(k-j)\prod_{j=k+1}^{m}(j-k) = \lim_{m\to\infty}\frac{m!}{m^k(-1)^k k!(m-k)!} \\
&= \lim_{m\to\infty}\frac{m(m-1)\cdots(m-k+1)(m-k)!}{m^k(-1)^k k!(m-k)!} = \lim_{m\to\infty}\frac{(-1)^k}{k!}[1+\frac{p(m)}{m^k}] = \frac{(-1)^k}{k!},
\end{aligned}$$

where, $p(m)$ is a polynomial in m of degree strictly less than k (see [14] for another way of calculating the same residues based on the recurrence relation $\Gamma(z+1) = z\Gamma(z)$).
Therefore, we obtain,

$$\frac{1}{2\pi i}\oint_C \Gamma(z)dz = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1} \Rightarrow \oint_C \Gamma(z)dz = \frac{2\pi i}{e}. \tag{11}$$

The result shown in (11) states another curious and interesting relationship between the imaginary unit i and the irrational numbers π and e, similar to the more familiar relation, $e^{i\pi} = -1$ and the real number $i^i = 1/\sqrt{e^\pi}$, both derived from Euler's identity. Now we turn our attention on finding the inverse transform of $\Gamma(z)$ using the residues method using again the contour depicted in Fig. 2. Specifically, we have that,

$$\gamma(n) = Z^{-1}\{\Gamma(z)\} = \frac{1}{2\pi i}\oint_C \Gamma(z) z^{n-1} dz$$
$$= \frac{1}{2\pi i}\oint_C [\lim_{m\to\infty}\Gamma_m(z)] z^{n-1} dz = \sum_{k=0}^{\infty} \text{res}_{z=-k}[\lim_{m\to\infty}\Gamma_m(z) z^{n-1}].$$

As can be seen from the last expression, the computation of the residues must be carried in more detail since each of them *depends* on n. We first evaluate explicitly the residues for $k = 0,1,2,3$ and then generalize inductively for any k. Since $z = -k$, then $z = 0,-1,-2,-3$. If $z = 0$, we have,

$$\text{res}_{z=0}[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}] = \text{res}_{z=0}[\lim_{m\to\infty} m!m^z z^n / z^2 \prod_{j=1}^{m}(z+j)] = \lim_{m\to\infty}\frac{d}{dz}(m!m^z z^n / \prod_{j=1}^{m}(z+j))\Big|_{z=0}.$$

The subexpression containing the derivative develops into,

$$\frac{d}{dz}\left(\frac{m!m^z z^n}{\prod_{j=1}^{m}(z+j)}\right) = \frac{\prod_{j=1}^{m}(z+j)[m!m^z z^n]' - m!m^z z^n[\prod_{j=1}^{m}(z+j)]'}{[\prod_{j=1}^{m}(z+j)]^2}$$
$$= \frac{m!m^z nz^{n-1} + m!m^z z^n \ln(m)}{\prod_{j=1}^{m}(z+j)} - \frac{m!m^z z^n \sum_{j=1}^{m}\prod_{\ell\neq j}(z+\ell)}{[\prod_{j=1}^{m}(z+j)]^2}. \quad (12)$$

If $n = 0$, the first term simplifies to, $m!\ln(m)/\prod_{j=1}^{m} j = m!\ln(m)/m! = \ln(m)$, and the second term is reduced to

$$\frac{m!\sum_{j=1}^{m}\prod_{\ell\neq j}\ell}{[\prod_{j=1}^{m} j]^2} = \frac{\sum_{j=1}^{m}\prod_{\ell\neq j}\ell}{\prod_{j=1}^{m} j} = \sum_{j=1}^{m}\frac{\prod_{\ell\neq j}\ell}{\prod_{\ell=1}^{m}\ell} = \sum_{j=1}^{m}\frac{1}{j}.$$

Combining the two terms, the corresponding residue value at $z = 0$ for $n = 0$ gives the negative of the Euler-Mascheroni constant, i. e.,

$$\lim_{m\to\infty}(\ln m - \sum_{j=1}^{m}\frac{1}{j}) = -\lim_{m\to\infty}(\sum_{j=1}^{m}\frac{1}{j} - \ln m) = -\gamma = -0.577215664901532\cdots$$

On the other hand, for $n = 1$, the first term of (12) simplifies to $m!/m! = 1$ and the second term equals zero. The residue value at $z = 0$ for $n = 1$ is 1 and it should be clear that if $n > 1$ both terms of (12) are zero. Therefore, for any $n \in \mathbb{N}$, we can write,

$$\text{res}_{z=0}[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}] = -\gamma\delta(n) + \delta(n-1). \quad (13)$$

We continue with the evaluation of the residues produced by the poles of $\Gamma(z)$ located at $z = -1,-2,-3$.

The details follow,

$$\operatorname{res}_{z=-1}[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}] = \lim_{m\to\infty}[m!m^z z^n / z^2 \prod_{j\neq 1}(z+j)|_{z=-1}]$$

$$= \lim_{m\to\infty}\frac{m!m^{-1}(-1)^n}{(-1)^2 1\cdots(m-1)} = (-1)^n,$$

$$\operatorname{res}_{z=-2}[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}] = \lim_{m\to\infty}[m!m^z z^n / z^2 \prod_{j\neq 2}(z+j)|_{z=-2}]$$

$$= \lim_{m\to\infty}\frac{m!m^{-2}(-2)^n}{(-2)^2(-1)1\cdots(m-2)} = \frac{(-2)^{n-1}}{2},$$

$$\operatorname{res}_{z=-3}[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}] = \lim_{m\to\infty}[m!m^z z^n / z^2 \prod_{j\neq 3}(z+j)|_{z=-3}]$$

$$= \lim_{m\to\infty}\frac{m!m^{-3}(-3)^n}{(-3)^2(-2)(-1)1\cdots(m-3)} = -\frac{(-3)^{n-1}}{6}.$$

By induction, we can generalize the previous pattern for any $k > 0$. Thus, we have

$$\lim_{m\to\infty}\frac{m!m^{-k}(-k)^n}{(-k)^2\prod_{j\neq k}(j-k)} = \lim_{m\to\infty}\frac{m!m^{-k}(-k)^{n-1}}{(-k)\prod_{j=1}^{k-1}(-1)(k-j)\prod_{j=k+1}^{m}(j-k)}$$

$$= \lim_{m\to\infty}\frac{m(m-1)(m-2)\cdots(m-k+1)(m-k)!(-k)^{n-1}}{m^k(-1)^k k!(m-k)!}$$

$$= \lim_{m\to\infty}\frac{[m^k + p(m)](-1)^{-k}(-k)^{n-1}}{m^k k!} = \frac{(-1)^k(-k)^{n-1}}{k!},$$

where the last equality follows from the fact that the degree of the polynomial in m, denoted by $p(m)$, is strictly less than k; also, recall that $(-1)^{-k} = (-1)^k$ for any $k \in \mathbb{Z}^+$. Finally, by summing all residues we find the resulting expression of the inverse Z-transform of the gamma function, i. e.,

$$\gamma(n) = -\gamma\delta(n) + \delta(n-1) + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{n-1}}{k!}u(n). \qquad (14)$$

The elements of the gamma sequence for $n = 0$ to $n = 4$ are evaluated by substitution of each n in (14) as shown next:

$$\gamma(0) = -\gamma + 0 + (-1)^{0-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{0-1}}{k!} = -\gamma - \sum_{k=1}^{\infty}\frac{(-1)^k}{kk!} = -\gamma + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{kk!},$$

$$\gamma(1) = 0 + 1 + (-1)^{1-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{1-1}}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1},$$

$$\gamma(2) = 0 + 0 + (-1)^{2-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{2-1}}{k!} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!} = e^{-1},$$

$$\gamma(3) = 0 + 0 + (-1)^{3-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{3-1}}{k!} = \sum_{k=1}^{\infty}\frac{(-1)^k k}{(k-1)!} = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell+1}(\ell+1)}{\ell!}$$

$$= -\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\ell+1)}{\ell!} = -\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\ell}{\ell!} - \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!} = \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{(\ell-1)!} - e^{-1} = 0,$$

$$\gamma(4)=0+0+(-1)^{4-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{4-1}}{k!}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^2}{(k-1)!}=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\ell+1)^2}{\ell!}$$
$$=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}(\ell^2+2\ell+1)=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\ell^2}{\ell!}+2\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\ell}{\ell!}+\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}$$
$$=\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}\ell}{(\ell-1)!}+2\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}}{(\ell-1)!}+e^{-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(k+1)}{k!}+2\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}+e^{-1}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}k}{k!}-\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}-2\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}+e^{-1}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}-2e^{-1}=-e^{-1}.$$

Note that computation of $\gamma(3)$ requires a single change of summation index ($\ell=k-1$ in the 3rd equality) whereas $\gamma(4)$ needs a couple of changes of summation index ($\ell=k-1$ in the 3rd equality and $k=\ell-1$ in the 7th equality). The values of the gamma sequence for $n\in\{5,6,7,8,9,10\}$ can also be determined as shown previously but the details are not explicitly given since the algebraic work is more cumbersome to follow. However, the resulting values are given by $\gamma(5)=e^{-1}$, $\gamma(6)=2e^{-1}$, $\gamma(7)=-9e^{-1}$, $\gamma(8)=9e^{-1}$, $\gamma(9)=50e^{-1}$, and $\gamma(10)=267e^{-1}$. Looking carefully to these numerical results as well as the steps that manipulate algebraically specific summations, equation (14) can be written in a more compact form as follows,

$$\gamma(n)=-\gamma\delta(n)+\delta(n-1)+e^{-1}\alpha_n u(n) \quad \text{where} \quad e^{-1}\alpha_n=(-1)^{n-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{n-1}}{k!}. \tag{15}$$

For the sake of completeness, Table 1 lists the numerical values of the gamma sequence for $n=0$ to $n=21$. Clearly, the gamma sequence (as many others) is an *increasing* sequence and for values of $n\geq 13$ it grows quite fast. However, we remark that the *infinite series* appearing in the last term of (14), or equivalently the term $e^{-1}\alpha_n$ in (15), *converges* for any $n\geq 0$. Note that the α_n values are integers (last column of Table 1). Symbolic computation of the infinite series (last term in (14)) was realized using the Maple software [17] and numerical verification was done by approximating the same series with a finite sum of 100 terms using the Decimal Basic programming language [18].

Table 1. Numerical values of the γ sequence and associated integer multiple α_n of e^{-1}.

n	$\gamma(n)$	α_n	n	$\gamma(n)$	α_n
0	0.2193839	0	11	151.9342092	413
1	0.3678794	1	12	801.9771818	2180
2	0.3678794	1	13	−6522.8703714	−17731
3	0.0000000	0	14	18590.0518007	50533
4	−0.3678794	−1	15	40531.4853106	110176
5	0.3678794	1	16	−723544.1812567	−1996797
6	0.7357589	2	17	3656231.9977064	9938669
7	−3.3109150	−9	18	−3178006.7503211	−8638718
8	3.3109150	9	19	−102445249.8212360	−278475061
9	18.3939721	50	20	934765660.5768400	2540956509
10	−98.2238108	−267	21	-3611421102.5222800	-9816860358

4. Conclusion

In this research work we have find the gamma sequence of real numbers obtained by calculating the inverse Z-transform of the complex gamma function. In the context of discrete mathematical transforms the result here obtained adds a new transform pair to the list of well known Z-transform pairs. Future work will address the problem of finding an *explicit series* for α_n by splitting the constant factor e^{-1} (itself and infinite series) from the product $e^{-1}\alpha_n$ using Cauchy's rule for infinite series multiplication.

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