HIGHER DIMENSIONAL SPHERICAL SYMMETRIC UNIVERSE WITH POLYTROPIC EQUATION OF STATE

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Abstract
In this paper we have studied higher dimensional static spherically symmetric universe. The Einstein’s field equations with polytropic equation of state have been solved to obtain two classes of exact models which satisfy all the major features expected in a realistic stars. Physical analysis of the models have been carried out. It is interesting to note that our results resemble with the results obtained by Thirukkanesh and Ragel [1].

Keywords: Einstein’s Field Equations; Polytropic Equation of State; Spherical Symmetric Universe; Higher Dimensions.

1. Introduction
The current accelerated expansion of the universe has been verified by various resources like supernovae Ia (SNeIa) experiments (Riess et al.[2], Perlmutter et al. [3]), cosmic microwave background (CMB) radiations (Spergelet al. [4,5]), WMAP (Bennett et al. [6]), weak lensing (Jain & Taylor[7]), large scale structures LSS (Tegmark et al. [8]) etc. A dark fluid with high negative pressure, i.e. “Dark energy”, is assumed to be the reason behind the accelerated expansion of the universe. This dark fluid is said to occupy about 68.3% of our universe, by PLANCK 2013 (P. A. R. Adeet al. [9]) results, dominating the other components of the universe [i.e. dark matter component (26.8%) and baryonic matter (4.9%)]. Noteworthy revelations from the study of super strings and super gravity theories have established the importance of studying the universe with higher dimensions. A study by Weinberg [10] explains that the space-time should be different from four dimensions. As the perception of more than four dimensions is not abstract, string theories have been studied in 10 dimensions or 26 dimensions inspiring researchers to do the
same. Reddy and Venkateswara [11], Adhav et al.[12-16] have contributed to the study of higher dimensional cosmological models in general relativity and alternate theories of gravitation.

A rather challenging question among the researchers has been the description of compact astrophysical objects. Certain experimental observations [17-21] have shown that the standard neutron star models are not compatible in describing some of the compact objects of the universe. The description of various astrophysical phenomena like gravitational collapse and evolution of compact stars under various conditions is an area of research which is gaining interest. The physics of very high density matter is not yet very clear and many of the strange star studies have been performed within the framework of the bag model [22-25]. Detailed studies about the relativistic compact objects with charged [26-28] and uncharged [29, 30] anisotropic matter within the framework of MIT bag model have been done. Although this has been briefly studied with the linear equation of state, some work has been performed with non-linear equation of state by Varela et al. [31] and Feroze and Siddiqui [32] on anisotropic matter in the presence of electromagnetic field. Moreover, polytropic model with equation of state, \( p = k \rho^\gamma \) which was previously studied under the Newtonian gravity [33] and then later extended under general relativity [34, 35], is considered to be stiffer than the conventional bag model, but regarded to be valuable because it could help modelling stars composed of realistic matter, such as ideal gas, photon gas, degenerate Fermi gas and in particular quark matter.

In this paper, section 2 deals with the conversion of Einstein’s field equations to a new system of equations using a coordinate transformation. These equations are further simplified using the polytropic equation of state. In section 3, we obtain two classes of exact solutions to the Einstein’s field equations. The physical conditions that should be satisfied by a realistic relativistic star are discussed in section 4. Concluding remarks are included in section 5.

2. The field equations

Here we consider the five dimensional static spherically symmetric space-time. This consideration is in league with the models used for analyzing the relativistic astrophysical objects and their physical behaviors. Thus, the interior of a static spherically symmetric star is given by the model

\[
ds^2 = -u(r)dt^2 + v(r)dr^2 + r^2(d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\phi^2)
\]

in Schwarzschild coordinates \((x^\alpha) = (t, r, \theta_1, \theta_2, \phi)\).

The energy momentum tensor for an anisotropic neutral imperfect fluid sphere can take the form

\[T_{\alpha\beta} = diag(-\rho, p_r, p_r, p_r, p_r),\]

(2)

where \(\rho\) is the energy density, \(p_r\) is the radial pressure and \(p_\parallel\) is the tangential pressure. These quantities are measured relative to the co-moving fluid 4-velocity \(u^\alpha = e^{-\nu} \delta^\alpha_0\).

For the line element (1) and matter distribution (2), the Einstein field equations can be expressed as

\[
\frac{3}{r^2} \left[ r \left(1 - \frac{1}{2v} \right) - \frac{1}{2v} \right] = \rho
\]

(3)

\[
-\frac{3}{r^2} \left(1 - \frac{1}{v} \right) + \frac{3u'}{2uvr} = p_r
\]

(4)

\[
\frac{u''}{2uv} - \frac{u'v'}{4uv^2} - \frac{u'^2}{4u^2v} + \frac{u'}{uur} - \frac{v'}{vr^2} + \frac{1}{r^2} = p_r
\]

(5)

where primes denote differentiation with respect to \(r\).
The field equations (3)-(5) are taken in natural limits $[8\pi G/c^4] = 1$ & $c = 1$. These equations govern the behavior of the gravitational field for an anisotropic imperfect fluid. The mass contained within a radius $r$ of the sphere is defined as

$$m(r) = \frac{1}{2} \int_0^r \omega^2 \rho(\omega) d\omega.$$  

An alternate form of the field equations can be obtained using the transformations

$$x = Cr^2, Z(x) = \frac{1}{v}, A^2 y^2 (x) = u,$$  

where $A$ and $C$ are arbitrary constants. Durgapal and Bannerji [36] have first introduced this transformation in their study. Using the transformation (7), the equations (3)-(5) become

$$3\left(1 - \frac{Z}{x} - \frac{\dot{Z}}{x}\right) = \frac{\rho}{C},$$  

$$3\left(\frac{Z}{x} - 1 + 2 \frac{\dot{Z}}{y} + \frac{\ddot{Z}}{y}\right) = \frac{p_r}{C},$$  

$$4xz \frac{\ddot{y}}{y} + (2xz + 4Z) \frac{\dot{y}}{y} + 2z + \frac{Z-1}{x} = \frac{p_t}{C},$$

where dots denote differentiation with respect to the variable $x$. The mass function (6) becomes

$$m(x) = \frac{1}{4C^{3/2}} \int_0^x \sqrt{w} \rho(w) dw,$$

in terms of the new variables in (7).

For a realistic relativistic star, the expected matter distribution should satisfy a barotropic equation of state $p_r = p_r(\rho)$. In this paper, we have considered the polytropic equation of state

$$p_r = k \rho^{1+1/n},$$

where $k$ is the polytropic constant and $n$ is the polytropic index. Now the system of equations (8)-(10) can be written as

$$\frac{\rho}{C} = 3\left(1 - \frac{Z}{x} - \frac{\dot{Z}}{x}\right),$$

$$p_r = k \rho^{1+1/n},$$

$$p_t = p_r + \Delta,$$

$$\frac{\Delta}{C} = 4xz \frac{\ddot{y}}{y} + (2xz - 2z) \frac{\dot{y}}{y} + 2z + \frac{2(1-Z)}{x},$$

$$\frac{\dot{y}}{y} = \frac{3^{1/n} k C^{3/2} (1 - \frac{Z}{x} - \frac{\dot{Z}}{x})^{1+1/n} + 1 - \frac{Z}{2xz}}{2Z},$$

where the quantity $\Delta = p_t - p_r$ is the measure of anisotropy in this model.
3. Exact models
The system of equations (13)-(17) contain six independent variables with only five independent equations. Thus, we can specify one of the quantities involved in the integration process. Equation (17) is the main equation in this integration process. In this case, we specify the gravitational potential $Z$ so that it is possible to integrate equation (17). The explicit solution of the Einstein system (13)–(17) then follows. We make a particular choice

$$Z = (1 - ax)^2,$$  \hspace{1cm} (18)

where $a$ is a real constant.

The gravitational potential $Z$ is regular at the origin and well behaved in the stellar interior for a wide range of values for the parameters $a$. Similar form of gravitational potential were previously used to study a charged perfect fluid source[34]. Therefore, the forms chosen in (18) is physically reasonable. By substituting (18) into (17) we get

$$\frac{\dot{y}}{y} = \frac{a(2 - ax)}{2(1 - ax)} + \frac{3^{1/n}kC^{n/a}a^{1/(n/a)}(4 - 3ax)^{1/(1/n)}}{2(1 - ax)^2}.$$  \hspace{1cm} (19)

If the values of the polytropic index $n$ are specified, equation (19) can be integrated giving solution for the system of equations. We shall consider the following two cases of physical interest.

3.1 The case $n = 1$:
When $n = 1$, equation (19) becomes

$$\frac{\dot{y}}{y} = \frac{a(2 - ax)}{2(1 - ax)^2} + \frac{3Ca^2(4 - 3ax)^2}{2(1 - ax)^2}.$$  \hspace{1cm} (20)

On integrating (20), we obtain

$$y = d_i(1 - ax)^{(1 + 18aCk)/2} \exp\left[\frac{1 + 3aCk(1 - 9(1 - ax)^2)}{2(1 - ax)}\right],$$  \hspace{1cm} (21)

where $d_i$ is the constant of integration.

Hence an exact model for the system (13)-(17) is as follows:

$$v = \frac{1}{(1 - ax)^2},$$  \hspace{1cm} (22)

$$u = A^2 d_i^2 (1 - ax)^{(1 + 18aCk)/2} \exp\left[\frac{1 + 3aCk(1 - 9(1 - ax)^2)}{(1 - ax)}\right],$$  \hspace{1cm} (23)

$$\rho = 3aC(4 - 3ax),$$  \hspace{1cm} (24)

$$p_r = k\rho^2,$$  \hspace{1cm} (25)

$$p_t = p_r + \Delta.$$  \hspace{1cm} (26)

The solutions (22)-(27) are given in simple elementary function so that it may be used to model an anisotropic star with quadratic equation of state.
3.2 The case \( n = 2 \):

When \( n = 2 \), equation (19) becomes

\[
\frac{\dot{y}}{y} = \frac{a(2-ax)}{2(1-ax)} + \frac{k\sqrt{3}Ca^{3/2}}{2(1-ax)^2}(4-3ax)^{3/2},
\]

(28)

On integrating (28), we obtain

\[
y = d_z \exp \left[ -\frac{6a^{3/2}kx\sqrt{12-9ax}}{2(ax-1)} + \frac{5\sqrt{3}ak\sqrt{4-3ax}}{2(ax-1)} + \frac{9\sqrt{3}ak\tanh^{-1}(\sqrt{4-3ax}) - \log(ax-1)}{2} - \frac{1}{2(ax-1)} \right].
\]

(29)

where \( d_z \) is the constant of integration.

Hence an exact model for the system (13)-(17) is as follows:

\[
v = \frac{1}{(1-ax)^2},
\]

(30)

\[
u = A^2 d_z^2 \exp \left[ -\frac{6a^{3/2}kx\sqrt{12-9ax}}{(ax-1)} + \frac{5\sqrt{3}ak\sqrt{4-3ax}}{(ax-1)} + \frac{9\sqrt{3}ak\tanh^{-1}(\sqrt{4-3ax}) - \log(ax-1)}{(ax-1)} \right],
\]

(31)

\[
\rho = 3aC(4-3ax),
\]

(32)

\[
p_r = k\rho^{3/2},
\]

(33)

\[
p_t = p_r + \Delta.
\]

(34)

The solutions (30)-(35) also are given in simple elementary function so that it may be used to model a polytropic star.

For both cases the mass function takes the form

\[
m(x) = \frac{ax^{3/2}(2-ax)}{2C}.
\]

(36)

4. Physical analysis:

Now as per Delgaty & Lake[37], the realistic star should satisfy following physical properties:

(i) regularity of the gravitational potentials at the origin;

(ii) positive definiteness of the energy density and the radial pressure at the origin;

(iii) vanishing of the pressure at some finite radius;

(iv) monotonic decrease of the energy density and the radial pressure with increasing radius.

(v) the interior metric match smoothly with the Schwarzschild exterior metric:

\[
ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

across the boundary \( r = R \), where \( M \) is the total mass of the sphere.

It is to be noted that all above physical properties for the case \( n = 1 \) and case \( n = 2 \) are satisfied by the models obtained in subsections [3.1] and [3.2] respectively.
5. Discussion
In this paper we have studied higher dimensional anisotropic static spherically symmetric star with polytropic equation of state. It has been observed that all the necessary physical properties of a realistic star are satisfied by the resulting two models in higher dimensions too. An attempt for retaining the results of Thirukkanesh and Ragel [1] has been made.

References


